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AN ASYMPTOTIC ANALYSIS OF  
SINGLE-JUNCTION SEMICONDUCTOR DEVICES

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## SIGNIFICANCE AND EXPLANATION

We present an analysis of a one-dimensional model of single-junction semiconductor devices (pn-junctions and certain resistors) when an external voltage is applied to the contacts. The model has the form of a system of six highly nonlinear first order ordinary differential equations subject to boundary conditions at the contacts of the device. The system is singularly perturbed (the derivatives of some components are multiplied by a small constant, the so called singular perturbation parameter). The dependent variables are the electrostatic potential, the hole and electron densities and the hole and electron current densities. A region of fast variation in the electrostatic potential and in the carrier distributions occurs due to the singular perturbation character of the problem. This region is in the interior of the device (internal layer) and represents the junction between differently doped areas. We derive formal asymptotic expansions of solutions as the singular perturbation parameter tends to zero and we prove that such an expansion 'represents' a solution. We also investigate the dependence of the total current on the externally applied voltage (voltage-current characteristic).

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ABSTRACT

In this paper we present an analysis of the fundamental one-dimensional semiconductor equations describing potential, carrier, and current density distributions in single-junction semiconductor devices when an external voltage is applied to the contacts. We reformulate the model equations by appropriate scaling as a singularly perturbed two point boundary value problem for a system of nonlinear ordinary differential equations. The right-hand side of the system has a jump discontinuity with respect to the independent variable (space-coordinate) representing the junction between differently doped sides of the device. The solution components are assumed to be continuous across this junction.

We give an existence proof for the reduced problem (the singular perturbation parameter is set to zero). The discontinuity of the right-hand side of the system produces a discontinuity in the reduced potential and reduced carrier distributions. This creates an internal layer in the corresponding solution components of the singularly perturbed problem. The current distributions have no internal layer. We also derive the (internal) layer equations and give an existence proof. No boundary layers occur.

We show that formal expansions actually represent (asymptotically) solutions of the singularly perturbed problem if the applied voltage is sufficiently small, and we investigate the dependence of the total current on the applied voltage. Numerical computations are reported.

AMS (MOS) Subject Classifications: 34C11, 34D15, 34E15

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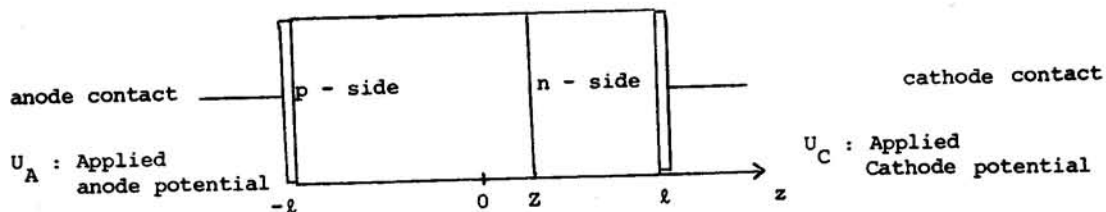
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# AN ASYMPTOTIC ANALYSIS OF SINGLE-JUNCTION SEMICONDUCTOR DEVICES

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## 1. Introduction

In this paper we present an analysis of a class of systems of ordinary differential equations, subject to boundary conditions, modelling pn-junction devices. The physical situation is as follows. A semiconductor (for example Silicon) is doped with acceptor atoms (negative ions) in the left side, with donor atoms (positive ions) in the right hand side and a bias  $U = U_A - U_C$  is applied to the contacts:



The device is assumed to have characteristic length  $2l (\approx 5 \times 10^{-3} \text{ cm})$  and the junction is at  $z = z_e (-l, l)$  (the term junction refers to the boundary of the n and p regions as well as to the whole device). The device is forward biased for  $U > 0$  and reverse biased for  $U < 0$ . The physics of pn-junction is explained in Sze (1969), Ashcroft and (1976) and R. A. Smith (1978).

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The equations describing the electrostatic potential, the carrier densities and the current densities within the device (in the static, one-dimensional case) are:

$$\begin{aligned}
 (1.1) \quad (a) \quad \psi'' &= \frac{q}{\epsilon} (n - p - (N_D^+(z) - N_A^-(z))) \text{ Poisson's equation} \\
 (b) \quad n' &= \frac{\mu_n}{D_n} n \psi' + \frac{1}{q D_n} J_n \text{ electron current relation} \\
 (c) \quad p' &= \frac{\mu_p}{D_p} p \psi' - \frac{1}{q D_p} J_p \text{ hole current relation} \\
 (d) \quad J_n' &= q R(n, p, J_n, J_p) \text{ continuity equation for electrons} \\
 (e) \quad J_p' &= q R(n, p, J_n, J_p) \text{ continuity equation for holes}
 \end{aligned}$$

for  $z \in [-l, l]$  ("'" denotes differentiation with respect to  $z$ ) subject to the boundary conditions

$$\begin{aligned}
 (1.2) \quad (a) \quad \psi(-l) &= U_T \ln \frac{n_i}{p(-l)} + U_A \text{ (anode)} \\
 (b) \quad \psi(l) &= U_T \ln \frac{n(l)}{n_i} + U_C \text{ (cathode)} \\
 (c) \quad n(\pm l)p(\pm l) &= n_i^2 \\
 (d) \quad n(\pm l) - p(\pm l) &= N_D^+(\pm l) - N_A^-(\pm l) .
 \end{aligned}$$

(See Van Roosbroeck (1950).)

The dependent variables (with units) in (1.1), (1.2) are

$$\begin{aligned}
 \psi &: \text{electrostatic potential (V)} \\
 \psi' &: \text{electrostatic field (Vcm}^{-1}\text{)} \\
 n &: \text{electron density (cm}^{-2}\text{)} \\
 p &: \text{hole density (cm}^{-3}\text{)} \\
 J_n &: \text{electron current density (A/cm}^2\text{)} \\
 J_p &: \text{hole current density (A/cm}^2\text{)} .
 \end{aligned}$$

All parameters in (1.1), (1.2) (except  $N_A^-(z)$ ,  $N_D^+(z)$ ) and the temperature  $T$  are assumed



to be constant. Table 1 gives the physical meaning and approximate numerical values of these parameters at  $T \approx 300K$  (room temperature) for silicon.

Table 1: Parameters for Silicon at  $T \approx 300K$

Parameter	Physical Meaning	Numerical Value
$q$	elementary charge	$10^{-19} \text{ As}$
$\epsilon$	permittivity constant	$10^{-12} \text{ As/Vcm}$
$\mu_n$	electron mobility	$10^3 \text{ cm}^2/\text{Vs}$
$\mu_p$	hole mobility	$10^3 \text{ cm}^2/\text{Vs}$
$D_n$	electron diffusion constant	$25 \text{ cm}^2/\text{s}$
$D_p$	hole diffusion constant	$25 \text{ cm}^2/\text{s}$
$n_i$	intrinsic number	$10^{10} \text{ cm}^{-3}$
$U_T = \frac{D_n}{\mu_n} = \frac{D_p}{\mu_p}$	thermal voltage	$0.025V$

$N_A^-$  is the density of electrically active acceptor atoms and  $N_D^+$  is the density of electrically active donor atoms and

$$(1.3) \quad C(z) = N_D^+(z) - N_A^-(z) \quad (\text{cm}^{-3})$$

is called doping (or impurity) profile. For the pn-junction  $C(z)$  is negative for  $z \in [-l, Z]$  (p-side) and positive for  $z \in (Z, l]$  (n-side) and is assumed to have a jump-discontinuity at  $z = Z$  (abrupt junction). We also investigate the less important and much simpler case  $C(z) > 0$  in  $[-l, l]$  (but still with a jump-discontinuity at  $z = Z$ ). These devices are called  $n^+n^-$  or  $nn^+$ -junctions (depending on whether

$C(z_1) > C(z_2)$  or  $C(z_1) < C(z_2)$  for all  $z_1 \in [-l, X]$ ,  $z_2 \in (X, l]$ ).

The analysis of  $p^+p$  and  $pp^+$  junctions ( $C(z) < 0$  on  $[-l, l]$ ) is analogous to the analysis of  $n^+n$  and  $nn^+$  junctions. Only  $n$  and  $p$ ,  $J_n$  and  $J_p$  have to be interchanged and  $\psi$  has to be substituted by  $-\psi$ .

The scalar function  $R \in C([0, \infty)^2 \times \mathbb{R}^2)$  in (1.1)(e) is called recombination term, it is the rate at which electron-hole carrier pairs are generated ( $R < 0$ ) or recombine

(vanish) ( $R > 0$ ),  $R(n,p,0,0) = 0$  for  $n, p$  such that  $p = n_1^2$  holds (equilibrium condition).

The Shockley-Read-Hall (SRH) recombination term

$$(1.4) \quad R = R_{SRH}(n,p) = \frac{np - n_1^2}{\tau_p(n+n_1) + \tau_n(p+n_1)} \text{ (cm}^{-3}\text{s}^{-1}\text{)}$$

describing thermal recombination, where  $\tau_n, \tau_p$  ( $\approx 10^{-6}$  s) are the electron and hole lifetimes, is widely used. Different ways of modelling  $R$  (which are necessary for very large  $|U|$ ) are given in Langer, Selberherr and Mader (1981) and Schütz, Selberherr and Pötzel (1982).

The boundary conditions (1.2)(c) express that the contacts  $z = \pm l$  are in thermal equilibrium and (1.2)(d) represents vanishing space charge at the contacts.

We only admit solutions of (1.1), (1.2) which fulfill

$$(1.5)(a) \quad \psi, n, p, J_n, J_p \in C^1([-l, l])$$

$$(1.5)(b) \quad \psi' \in C^1([-l, z]), \psi' \in C^1([z, l])$$

$$(1.6) \quad n > 0, p > 0 \text{ on } [-l, l].$$

(1.5) comes from the jump-discontinuity of  $C(z)$  ( $\psi''$  cannot be continuous if  $n, p$  are continuous), the equation (1.1)(a) has to be fulfilled for the right hand limit and for the left hand limit of  $\psi''$  at  $z = Z$ . (1.6) has to hold on physical grounds since  $n, p$  are densities.

In this paper we scale (1.1), (1.2) such that we obtain a singular perturbation problem. The perturbation parameter (called  $\lambda$  in the sequel) is equal to  $\frac{\lambda_D}{l}$  where  $\lambda_D = \left( \frac{\epsilon U_T}{q \max_{z \in [-l, l]} |C(z)|} \right)^{1/2}$  is the minimal Debye length.

We present an asymptotic analysis of (1.1), (1.2) (for  $\lambda$  small, which corresponds to large doping  $|C|$ ). The discontinuity of  $C(z)$  at  $z = Z$  produces an (internal) layer in the fast components  $\psi, \psi', n, p$ .  $J_n, J_p$  are the slow components (uniformly  $C^1$  as  $\lambda \rightarrow 0+$ ).

We derive the reduced problem ( $\lambda = 0$ ), the layer equations and give existence theorems for both. The reduced problem has the form of a two-point boundary value problem with interface conditions at the discontinuity. Using these results we prove an existence result for the full problem (1.1), (1.2) (for sufficiently large doping  $|C|$ , that means  $\lambda$  small) assuming that the recombination rate  $R \equiv 0$  (corresponding to infinite electron and hole lifetime) and that  $|U|$  is small. We show that (for  $\lambda$  sufficiently small) there is a solution of (1.1), (1.2) whose fast components are close to the sum of the (corresponding) 'reduced' solution components and the layer terms and whose slow components are close to the corresponding 'reduced' solution components. No layers at the contacts occur since the 'reduced' solution fulfills the 'reduced' boundary conditions.

We also investigate the dependence of the total current  $J = J_n + J_p$  on the applied voltage  $U$  ( $J$  is a constant because of (1.1)(d),(e)!). It turns out that  $J$  is asymptotically (as  $\lambda \rightarrow 0+$ ) a linear function of  $U$  if  $C(z) > 0$  on  $[-l, l]$  ( $n^+n$  and  $nn^+$  junctions are resistors) and  $J$  is asymptotically an exponential function of  $U$  if  $C$  changes sign at  $z$ .

The singular perturbation approach to pn-junction modelling was suggested by many authors. Vasilev'a and Butuzow (1978), Vasilev'a and Stelmakh (1977) and D. Smith (1980) investigated a much simplified model (they assume that the current densities are known instead of the applied voltage, that  $z = 0$  and that  $C(z)$  is odd) and prove an existence theorem using the asymptotic expansions. The authors of this paper analyzed (1.1), (1.2) (1982) under the (pretty unrealistic) assumption that the junction  $z$  is in the middle of the device and that the doping profile  $C$  is odd. This allows to reduce the internal layer problem to a boundary layer problem. The advantages of the singular perturbation approach for the numerical solution of (1.1), (1.2) is also explained in the latter paper.

The generalization of the presented theory to multilayer structures like bulk-barrier diodes (see Langer, Selberherr and Mader (1981)) or thyristors (see Sze (1969)) is straightforward.



This paper is organized as follows. In Section 2 we perform the scaling and reformulation of (1.1), (1.2) as a singular perturbation problem, in Section 3 we derive the expansions, prove existence theorems for the reduced problem and the internal layer problem and in Section 4 we give the existence proof for  $R \equiv 0$  the full singularly perturbed problem in the case  $\lambda$  and  $|U|$  small. Numerical results for large  $U$  are demonstrated in Section 5.

## 2. Scaling

We scale the dependent variables as follows

$$(2.1) \quad \psi_s = \frac{\psi}{U_T}, \quad n_s = \frac{n}{C}, \quad p_s = \frac{p}{C}$$

$$(2.2) \quad J_{n_s} = \frac{\ell J_n}{D_n q C}, \quad J_{p_s} = \frac{\ell J_p}{D_p q C}$$

where  $\bar{C} = \max_{z \in [-\ell, \ell]} |C(z)|$  and the independent variable

$$(2.3) \quad x = \frac{z}{\ell}.$$

Then (1.1) reads

$$(2.4) \quad \begin{aligned} (a) \quad & \lambda^2 \psi_s'' = n_s - p_s - D(x) \\ (b) \quad & n_s' = n_s \psi_s' + J_{n_s} \\ (c) \quad & p_s' = -p_s \psi_s' - J_{p_s} \\ (d) \quad & J_{n_s}' = \frac{\ell^2}{D_n \bar{C}} R(\bar{C} n_s, \bar{C} p_s, \frac{D_n q \bar{C}}{\ell} J_{n_s}, \frac{D_p q \bar{C}}{\ell} J_{p_s}) \\ (e) \quad & J_{p_s}' = -\frac{\ell^2}{D_p \bar{C}} R(\bar{C} n_s, \bar{C} p_s, \frac{D_p q \bar{C}}{\ell} J_{n_s}, \frac{D_p q \bar{C}}{\ell} J_{p_s}) \end{aligned}$$

for  $-1 \leq x \leq 1$  ("'" denotes now differentiation with respect to  $x$ ). We have set

$$(2.5) \quad D(x) = \frac{C(\ell x)}{\bar{C}}, \quad (|D(x)| \leq 1)$$

and

$$(2.6) \quad \lambda^2 = \frac{\lambda_D^2}{\ell^2} = \frac{\epsilon U_T}{\ell^2 q \bar{C}}.$$

The boundary conditions (2.2) are

$$(2.8) \quad \begin{aligned} (a) \quad & \psi_s(-1) = \ell n\left(\frac{\gamma \lambda^2}{p_s(-1)}\right) + \frac{U_A}{U_T} \\ (b) \quad & \psi_s(1) = \ell n\left(\frac{n_s(1)}{\gamma \lambda^2}\right) + \frac{U_C}{U_T} \\ (c) \quad & n_s(\pm 1) p_s(\pm 1) = \gamma^4 \lambda^4 \\ (d) \quad & n_s(\pm 1) - p_s(\pm 1) = D(\pm 1) \end{aligned}$$

where

$$(2.9) \quad \gamma^2 = \frac{\ell q n_i}{\epsilon U_T}$$

holds.

We now assume that the recombination term  $R$  is such that

$$(2.10) \quad R(\bar{C}_n, \bar{C}_p, \frac{D_n q \bar{C}}{\ell} J_{n_s}, \frac{D_p q \bar{C}}{\ell} J_{p_s}) = \bar{C} S(n_s, p_s, J_{n_s}, J_{p_s}, \gamma \lambda)$$

holds, where  $S \in C([0, \infty)^2 \times \mathbb{R}^2 \times [0, \infty))$  is independent of  $\bar{C}$ .

Dropping the index  $s$  the problem now reads

$$(2.11) \quad \left. \begin{aligned} (a) \quad \lambda^2 \psi'' &= n - p - D(x) \\ (b) \quad n' &= n \psi' + J_n \\ (c) \quad p' &= -p \psi' - J_p \\ (d) \quad J_n' &= S_n(n, p, J_n, J_p, \gamma \lambda) \\ (e) \quad J_p' &= -S_p(n, p, J_n, J_p, \gamma \lambda) \end{aligned} \right\} \quad -1 < x < 1$$

with  $S_n = \frac{\ell^2}{D_n} S$ ,  $S_p = \frac{\ell^2}{D_p} S$  subject to the boundary conditions

$$(2.12) \quad \begin{aligned} (a) \quad \psi(-1) &= \ell n \left( \frac{\gamma^2 \lambda^2}{p(-1)} \right) + \frac{U_A}{U_T} \\ (b) \quad \psi(1) &= \ell n \left( \frac{n(1)}{\gamma^2 \lambda^2} \right) + \frac{U_C}{U_T} \\ (c) \quad n(\pm 1) p(\pm 1) &= \gamma^4 \lambda^4 \\ (d) \quad n(\pm 1) - p(\pm 1) &= D(\pm 1) \end{aligned}$$

If  $D_n = D_p = D$  holds we have

$$(2.13) \quad S_n = S_p$$

Under this assumption and  $\tau_n = \tau_p = \tau$  we get for the SRH term with  $\beta = \frac{D\tau}{\ell^2}$

$$(2.14) \quad S_n = S_p = \frac{1}{\beta} \frac{np - \gamma^4 \lambda^4}{n + p + 2\gamma^2 \lambda^2}$$

Generally, the equilibrium condition implies that  $S_n(n, p, 0, 0, \gamma \lambda) = S_p(n, p, 0, 0, \gamma \lambda) = 0$

holds for  $n, p$  such that  $np = \gamma^4 \lambda^4$ .

The discontinuity of  $D$  occurs at

$$(2.15) \quad x = \frac{z}{\ell}$$

and the conditions on the solution of (2.11), (2.12) are

$$(2.16) \quad (a) \quad \psi, n, p, J_n, J_p \in C^1([-1, 1])$$

$$(2.16) \quad (b) \quad \psi' \in C^1([-1, x]), \quad \psi' \in C^1([x, 1])$$

$$(2.17) \quad n > 0, \quad p > 0 \quad \text{on} \quad [-1, 1]$$

(see (1.5), (1.6)).

The boundary values for  $n$ ,  $p$  and  $\psi$  can be computed from (2.12)(c),(d):

$$(2.18) \quad n(1) = \frac{1}{2} (D(1) + \sqrt{D(1)^2 + 4\gamma^4 \lambda^4}), \quad p(1) = \frac{1}{2} (-D(1) + \sqrt{D(1)^2 + 4\gamma^4 \lambda^4})$$

$$(2.19) \quad n(-1) = \frac{1}{2} (D(-1) + \sqrt{D(-1)^2 + 4\gamma^4 \lambda^4}), \quad p(-1) = \frac{1}{2} (-D(-1) + \sqrt{D(-1)^2 + 4\gamma^4 \lambda^4})$$

$$(2.20) \quad \psi(-1) = \ell n \left[ \frac{2\gamma^2 \lambda^2}{-D(-1) + \sqrt{D(-1)^2 + 4\gamma^4 \lambda^4}} \right] + \frac{U_A}{U_T}$$

$$(2.21) \quad \psi(1) = \ell n \left[ \frac{-D(1) + \sqrt{D(1)^2 + 4\gamma^4 \lambda^4}}{2\gamma^2 \lambda^2} \right] + \frac{U_C}{U_T}$$

For  $\lambda$  small the problem (2.11), (2.18), (2.19), (2.20), (2.21) constitutes a singularly perturbed two-point boundary value problem.

$\lambda$  small means that  $\bar{C}$  is large (assuming that  $\ell$  is constant). In practical cases  $\bar{C} > 10^{17}$  such that  $\lambda^2 < 0.4 \times 10^{-6}$  holds. For the following analysis we assume for the sake of simplicity that  $D(x)$  is independent of  $\lambda$  (it would suffice to assume that  $D$  is analytic in  $\lambda$ ), that means that the doping  $|C(z)|$  increases 'as a whole' as  $\lambda \rightarrow 0+$ . Actually, the asymptotic analysis presented in the next sections requires that

$$(2.22) \quad \frac{\min_{z \in [-\ell, \ell]} |D(z)|}{\max_{z \in [-\ell, \ell]} |D(z)|} \gg \lambda^2$$

and since  $\gamma^2 \lambda^2 = \frac{n_1}{\max_{z \in [-\ell, \ell]} |C(z)|}$  that

$$(2.23) \quad n_1 \ll \max_{z \in [-\ell, \ell]} |C(z)|$$

holds.

The two cases we deal with now are

$$(A) \quad \begin{aligned} D(x) < 0 \text{ for } x \in [-1, X]; D(x) > 0 \text{ for } x \in (X, 1] \text{ and} \\ |D(x)| > D_A \text{ for } x \in [-1, 1] \end{aligned}$$

which corresponds to the pn-junction and

$$(B) \quad D(x) > D_B > 0 \text{ for } x \in [-1, 1]$$

corresponding to an  $nn^+$  or  $n^+n$  junction. In both cases (A) and (B) we assume that  $D(X+)$ ,  $D(X-)$  (we use the notation of  $f(X\pm) = \lim_{x \rightarrow X\pm} f(x)$  in the sequel) exist and  $D(X+) \neq D(X-)$  and that  $D$  is sufficiently smooth everywhere else.

The analysis of the scaled problem is complicated by the logarithmic blow-up of the boundary data of  $\psi$  as given by (2.20), (2.21) in the case (A).

The potential difference of the contacts is given by

$$(2.24) \quad \psi(-1) - \psi(1) = \frac{U}{U_T} + \frac{U_{b1}(\lambda)}{U_T}$$

where the build-in-voltage  $U_{b1}(\lambda)$  (i.e. the voltage due to doping) is given by

$$(2.25) \quad \frac{U_{b1}(\lambda)}{U_T} = \ln \left[ \frac{D(-1) + \sqrt{D(-1)^2 + 4\gamma^4 \lambda^4}}{D(1) + \sqrt{D(1)^2 + 4\gamma^4 \lambda^4}} \right] = \begin{cases} \ln \left( \frac{\gamma^4 \lambda^4}{D(1)|D(-1)|} \right) + O(\gamma^4 \lambda^4), & D(-1) < 0 \text{ (A)} \\ \ln \left( \frac{D(-1)}{D(1)} \right) + O(\gamma^4 \lambda^4), & D(-1) > 0 \text{ (B)} \end{cases}$$

$U_{b1}(\lambda)$  is bounded as  $\lambda \rightarrow 0+$  in the case (B). Since (2.11) depends only on  $\psi'$ ,  $\psi''$  (and not on  $\psi$ ) we can therefore remove the singularity in the boundary conditions by substituting  $\psi$  by  $\psi - \ln \frac{1}{\gamma^2 \lambda^2}$ . The equations (2.11) remain unchanged and the new boundary conditions for the case (B) are:

$$(2.26) \quad \psi(-1) = \ln \left[ \frac{1}{2} (D(-1) + \sqrt{D(-1)^2 + 4\gamma^4 \lambda^4}) \right] + \frac{U_A}{U_T}$$

$$(2.27) \quad \psi(1) = \ln \left[ \frac{1}{2} (D(1) + \sqrt{D(1)^2 + 4\gamma^4 \lambda^4}) \right] + \frac{U_C}{U_T}$$

### 3. Expansions

In this Section we apply the standard approach for singularly perturbed boundary value problems to the semiconductor problem. We assume that the solution has a formal asymptotic expansion in  $\lambda$ , each term in the series being the sum of a uniformly smooth function and layer terms.

A problem that occurs is the blow up of the boundary values of  $\psi$  in the case (A), which implies that 'reduced' boundary conditions ( $\lambda=0$ ) cannot be defined formally for  $\psi$ . For the derivation of the expansions we set in the case (A)

$$(3.1)(a) \quad (a) \quad \psi(-1) = \psi_-, \quad (b) \quad \psi(1) = \psi_+$$

and assume that  $\psi_-, \psi_+$  are independent of  $\lambda$  (this will be justified later). In the case (B) 'reduced' boundary conditions for  $\psi$  can be obtained from (2.26), (2.27) and we set:

$$(3.1) \quad (c) \quad \psi_- = \ln D(-1) + \frac{U_A}{U_T}, \quad (d) \quad \psi_+ = \ln D(1) + \frac{U_C}{U_T}.$$

We make the following 'ansatz':

$$(3.2)(a) \quad \psi(x, \lambda) \sim \bar{\psi}(x) + \hat{\psi}(\sigma) + \tilde{\psi}_L(\tau) + \tilde{\psi}_R(\phi) + \dots$$

$$(3.2)(b) \quad n(x, \lambda) \sim \bar{n}(x) + \hat{n}(\sigma) + \tilde{n}_L(\tau) + \tilde{n}_R(\phi) + \dots$$

$$(3.2)(c) \quad p(x, \lambda) \sim \bar{p}(x) + \hat{p}(\sigma) + \tilde{p}_L(\tau) + \tilde{p}_R(\phi) + \dots$$

$$(3.2)(d) \quad J_n(x, \lambda) \sim \bar{J}_n(x) + \hat{J}_n(\sigma) + \tilde{J}_{n_L}(\tau) + \tilde{J}_{n_R}(\phi) + \dots$$

$$(3.2)(e) \quad J_p(x, \lambda) \sim \bar{J}_p(x) + \hat{J}_p(\sigma) + \tilde{J}_{p_L}(\tau) + \tilde{J}_{p_R}(\phi) + \dots$$

where the dots stand for a power series in  $\lambda$  (starting with the  $O(\lambda)$ -term) whose coefficients are of the same form as the given  $O(1)$  terms. The fast variables are

$$(3.3) \quad \left. \begin{aligned} (a) \quad \sigma &= \frac{x-X}{\lambda} \\ (b) \quad \tau &= \frac{x+1}{\lambda} \\ (c) \quad \phi &= \frac{x-1}{\lambda} \end{aligned} \right\} x \in [-1, 1]$$



The functions marked with '-' denote the reduced solution (of order zero), '^' denotes the internal layer terms (at  $x = X$ ) (of order zero), '~' denotes the layer terms (of order zero) decaying from the left boundary  $x = -1$  (with index  $l$ ) and the layer terms (of order zero) decaying from the right boundary  $x = +1$  (with index  $r$ ) resp. The boundary condition

$$(3.4) \quad (a) \quad \hat{\psi}(\pm\infty) = \hat{n}(\pm\infty) = \hat{p}(\pm\infty) = \hat{J}_n(\pm\infty) = \hat{J}_p(\pm\infty) = 0$$

$$(3.4) \quad (b) \quad \tilde{\psi}_l(\infty) = \tilde{n}_l(\infty) = \tilde{p}_l(\infty) = \tilde{J}_n(\infty) = \tilde{J}_p(\infty) = 0$$

$$(3.4) \quad (c) \quad \tilde{\psi}_r(-\infty) = \tilde{n}_r(-\infty) = \tilde{p}_r(-\infty) = \tilde{J}_n(-\infty) = \tilde{J}_p(-\infty) = 0$$

hold, since the internal layer terms are regarded as functions on  $R$ , the left layer terms are regarded as functions on  $[0, \infty)$ , the right layer terms as functions on  $(-\infty, 0]$ .

We assume that  $s_n, s_p \in C^1([0, \infty)^2 \times R^2 \times [0, \gamma\lambda_0])$  and that  $\lambda < \lambda_0$ . Inserting into (2.11), comparing 0(1)-terms and evaluating away from  $x = \pm 1, X$  gives the reduced problem (or order zero):

$$(3.5) \quad \left. \begin{aligned} (a) \quad 0 &= \bar{n} - \bar{p} - D(x) \\ (b) \quad \bar{n}' &= \bar{n} \bar{\psi}' + \bar{J}_n \\ (c) \quad \bar{p}' &= -\bar{p} \bar{\psi}' - \bar{J}_p \\ (d) \quad \bar{J}_n' &= s_n(\bar{n}, \bar{p}, \bar{J}_n, \bar{J}_p, 0) \\ (e) \quad \bar{J}_p' &= -s_p(\bar{n}, \bar{p}, \bar{J}_n, \bar{J}_p, 0) \end{aligned} \right\} \quad -1 < x < 1.$$

We have to expect that  $\bar{\psi}, \bar{n}, \bar{p}$  are discontinuous at  $x = X$ , therefore (3.5) has to hold for the right hand and left hand limits at  $x = X$ . Evaluation close to  $X+$  gives the right (zeroth order) (internal) layer problem

$$(3.6) \quad \left. \begin{aligned} (a) \quad \hat{\psi} &= \hat{n} - \hat{p} \\ (b) \quad \hat{n} &= (\hat{n} + \bar{n}(X+)) \hat{\psi} \\ (c) \quad \hat{p} &= -(\hat{p} + \bar{p}(X+)) \hat{\psi} \\ (d) \quad \hat{J}_n &= 0 \\ (e) \quad \hat{J}_p &= 0 \end{aligned} \right\} \quad 0 < \sigma < \infty$$

('·' denotes differentiation with respect to the corresponding fast variable in the sequel) and evaluation close to  $X^-$  gives the left (zeroth order) (internal) layer problem

$$(3.7) \quad \left. \begin{aligned} (a) \quad \ddot{\psi} &= \hat{n} - \hat{p} \\ (b) \quad \dot{\hat{n}} &= (\hat{n} + \bar{n}(X^-)) \dot{\psi} \\ (c) \quad \dot{\hat{p}} &= -(\hat{p} + \bar{p}(X^-)) \dot{\psi} \\ (d) \quad \dot{J}_n &= 0 \\ (e) \quad \dot{J}_p &= 0 \end{aligned} \right\} -\infty < \sigma < 0 .$$

Similarly we obtain the left (boundary) layer problem

$$(3.8) \quad \left. \begin{aligned} (a) \quad \ddot{\tilde{\psi}}_\ell &= \tilde{n}_\ell - \tilde{p}_\ell \\ (b) \quad \dot{\tilde{n}}_\ell &= (\tilde{n}_\ell + \bar{n}(-1)) \dot{\tilde{\psi}}_\ell \\ (c) \quad \dot{\tilde{p}}_\ell &= -(\tilde{p}_\ell + \bar{p}(-1)) \dot{\tilde{\psi}}_\ell \\ (d) \quad \dot{\tilde{J}}_{n_\ell} &= 0 \\ (e) \quad \dot{\tilde{J}}_{p_\ell} &= 0 \end{aligned} \right\} 0 < \tau < \infty .$$

The right (boundary) layer problem is obtained from (3.8) by substituting  $\bar{n}(-1), \bar{p}(-1)$  by  $\bar{n}(1), \bar{p}(1)$ .

Inserting into (2.19), (3.1)(a) and comparing  $O(1)$ -coefficients of  $\lambda$  gives the matching conditions at  $x = -1$

$$(3.9) \quad \begin{aligned} (a) \quad \bar{n}(-1) + \tilde{n}_\ell(0) &= \begin{cases} D(-1) , & D(-1) > 0 \\ 0 , & D(-1) < 0 \end{cases} \\ (b) \quad \bar{p}(-1) + \tilde{p}_\ell(0) &= \begin{cases} 0 , & D(-1) > 0 \\ -D(-1) , & D(-1) < 0 \end{cases} \\ (c) \quad \bar{\psi}(-1) + \tilde{\psi}_\ell(0) &= \psi_- \end{aligned}$$

(2.18), (3.1)(b) gives the matching conditions at  $x = 1$

$$\begin{aligned}
(3.10) \quad & (a) \quad \bar{n}(1) + \tilde{n}_r(0) = D(1) \\
& (b) \quad \bar{p}(1) + \tilde{p}_r(0) = 0 \\
& (c) \quad \bar{\psi}(1) + \tilde{\psi}_r(0) = \psi_+ .
\end{aligned}$$

The continuity conditions (2.16) give the interface conditions

$$\begin{aligned}
(3.11) \quad & (a) \quad \bar{n}(X-) + \hat{n}(0-) = \bar{n}(X+) + \hat{n}(0+) \\
& (b) \quad \bar{p}(X-) + \hat{p}(0-) = \bar{p}(X+) + \hat{p}(0+) \\
& (c) \quad \bar{\psi}(X-) + \hat{\psi}(0-) = \bar{\psi}(X+) + \hat{\psi}(0+) \\
& (d) \quad \hat{\psi}(0-) = \hat{\psi}(0+) \\
& (e) \quad \bar{J}_n(X-) + \hat{J}_n(0-) = \bar{J}_n(X+) + \hat{J}_n(0+) \\
& (f) \quad \bar{J}_p(X-) + \hat{J}_p(0-) = \bar{J}_p(X+) + \hat{J}_p(0+) .
\end{aligned}$$

From (3.7)(d),(e), (3.8)(d),(e) and from the analogous equations for the right  $J_n, J_p$ -layer terms we immediately conclude that

$$(3.12) \quad \hat{J}_n \equiv \hat{J}_p \equiv 0, \quad \tilde{J}_{n_\ell} \equiv \tilde{J}_{p_\ell} \equiv 0, \quad \tilde{J}_{n_r} \equiv \tilde{J}_{p_r} \equiv 0$$

since (3.4) has to hold. No zeroth order layers occur in  $J_n, J_p$ . The current densities are the slow components.

The problem (3.8), (3.9) has been dealt with in Markowich, Ringhofer, et al (1982) and it has been shown that

$$(3.13) \quad \tilde{\psi}_\ell \equiv \tilde{n}_\ell \equiv \tilde{p}_\ell \equiv 0$$

holds. The same analysis goes through for the right boundary layer terms and

$$(3.14) \quad \tilde{\psi}_r \equiv \tilde{n}_r \equiv \tilde{p}_r \equiv 0$$

follows. No zeroth order boundary layers occur, since the reduced boundary conditions for  $n, p$  ((2.18), (2.19) with  $\lambda = 0$ ) can be fulfilled by the reduced solution due to (3.5)(a).

By including more terms in the expansion (3.2) it turns out that higher order boundary layer terms occur. Similarly, higher order internal layers occur in the slow component  $J_n(J_p)$  if  $\frac{\partial S}{\partial n}$  or  $\frac{\partial S}{\partial p} \left( \frac{\partial S}{\partial n} \text{ or } \frac{\partial S}{\partial p} \right)$  are not constant zero.

Integrating (3.6)(b),(c), (3.7)(b),(c) gives

$$(3.15) \quad \hat{n}(\sigma) = \begin{cases} \bar{n}(x+)(e^{\hat{\psi}(\sigma)} - 1) & , \sigma > 0 \\ \bar{n}(x-)(e^{\hat{\psi}(\sigma)} - 1) & , \sigma < 0 \end{cases}$$

$$(3.16) \quad \hat{p}(\sigma) = \begin{cases} \bar{p}(x+)(e^{-\hat{\psi}(\sigma)} - 1) & , \sigma > 0 \\ \bar{p}(x-)(e^{-\hat{\psi}(\sigma)} - 1) & , \sigma < 0 \end{cases} .$$

Inserting (3.15), (3.16) into (3.11)(a),(b) gives

$$(3.17) \quad \bar{n}(x-)e^{\hat{\psi}(0-)} = \bar{n}(x+)e^{\hat{\psi}(0+)}$$

$$(3.18) \quad \bar{p}(x-)e^{-\hat{\psi}(0-)} = \bar{p}(x+)e^{-\hat{\psi}(0+)} .$$

From (3.11)(c), (3.5)(a) we get the interface conditions for the reduced problem

$$(3.19) \quad \begin{aligned} (a) \quad & \bar{n}(x+)e^{\bar{\psi}(x-)-\bar{\psi}(x+)} = \bar{n}(x-) \\ (b) \quad & (\bar{n}(x-) - D(x-))e^{\bar{\psi}(x-)-\bar{\psi}(x+)} = n(x+) - D(x+) \end{aligned}$$

and from (3.11)(e),(f) and (3.12)

$$(3.19) \quad \begin{aligned} (c) \quad & \bar{J}_n(x+) = \bar{J}_n(x-) \\ (d) \quad & \bar{J}_p(x+) = \bar{J}_p(x-) \end{aligned} .$$

The boundary conditions follow from (3.9), (3.10) and (3.13), (3.14)

$$(3.20) \quad \begin{aligned} (a) \quad & \bar{n}(-1) = \begin{cases} 0 & , D(-1) < 0 \\ D(-1) & , D(-1) > 0 \end{cases} \\ (b) \quad & \bar{\psi}(-1) = \psi_- \end{aligned}$$

$$(3.21) \quad \begin{aligned} (a) \quad & \bar{n}(1) = D(1) \\ (b) \quad & \bar{\psi}(1) = \psi_+ \end{aligned} .$$

Eliminating  $\bar{p}$  from (3.5)(b),(c) using (3.5)(a) gives the reduced equations

$$\begin{aligned}
(3.22) \quad & \left. \begin{aligned}
(a) \quad \bar{\psi}' &= \frac{D' - (\bar{J}_n + \bar{J}_p)}{2\bar{n} - D} \\
(b) \quad \bar{n}' &= \frac{(\bar{n}-D)\bar{J}_n - \bar{n}\bar{J}_p + \bar{n}D'}{2\bar{n} - D} \\
(c) \quad \bar{J}_n' &= S_n(\bar{n}, \bar{p}, \bar{J}_n, \bar{J}_p, 0) \\
(d) \quad \bar{J}_p' &= -S_p(\bar{n}, \bar{p}, \bar{J}_n, \bar{J}_p, 0)
\end{aligned} \right\} \begin{aligned}
& -1 \leq x < X \\
& \text{and} \\
& X < x \leq 1
\end{aligned}
\end{aligned}$$

assuming that  $D \in C^1([-1, X]) \cap C^1([X, 1])$ .  $\bar{p}$  is given by (3.5)(a):

$$(3.22) \quad (e) \quad \bar{p} = \bar{n} - D(x).$$

We obtain the internal layer problem by inserting (3.15), (3.16) into (3.6)(a), (3.7)(a):

$$\begin{aligned}
(3.23) \quad & (a) \quad \ddot{\psi} = \bar{n}(X-)e^{\hat{\psi}} - \bar{p}(X-)e^{-\hat{\psi}} - D(X-), \quad -\infty < \sigma < 0 \\
& (b) \quad \ddot{\psi} = \bar{n}(X+)e^{\hat{\psi}} - \bar{p}(X+)e^{-\hat{\psi}} - D(X+), \quad 0 < \sigma < \infty
\end{aligned}$$

subject to the boundary conditions

$$\begin{aligned}
(3.23) \quad & (c) \quad \hat{\psi}(-\infty) = 0 \\
& (d) \quad \hat{\psi}(\infty) = 0
\end{aligned}$$

and the interface conditions

$$\begin{aligned}
(3.23) \quad & (e) \quad \hat{\psi}(0+) - \hat{\psi}(0-) = \bar{\psi}(X-) - \bar{\psi}(X+) \\
& (f) \quad \dot{\hat{\psi}}(0+) = \dot{\hat{\psi}}(0-).
\end{aligned}$$

Because of (2.17) we require that the solutions  $\bar{n}, \bar{p}$  of the reduced problem are nonnegative, that means:

$$(3.24) \quad \bar{n}(x) > \max(0, D(x)), \quad x \in [-1, 1]$$

has to hold. Under this assumption we prove a simple consequence of the interface conditions (3.19).

**Lemma 3.1.** Assume that (3.24) holds (at least at  $X-, X+$ ). Then

$$\begin{aligned}
(3.25) \quad & D(X-) < 0, \quad D(X+) > 0 \implies \\
& (\bar{\psi}(X-) < \bar{\psi}(X+) \text{ and } \bar{n}(X+) > D(X+), \bar{n}(X-) > 0)
\end{aligned}$$

holds.

Proof: Assume first that  $\bar{\psi}(X-) = \bar{\psi}(X+)$ . Then (3.19)(a) implies that  $\bar{n}(X+) = \bar{n}(X-)$  and (3.19)(b) implies  $D(X-) = D(X+)$ . This is a contradiction to the assumption that  $D$  has a jump discontinuity at  $x = X$ . Therefore  $\bar{\psi}(X-) \neq \bar{\psi}(X+)$ . We compute  $\bar{n}(X+)$  from (3.19)

$$(3.26) \quad \bar{n}(X+) = \frac{D(X+) - D(X-) e^{\bar{\psi}(X-) - \bar{\psi}(X+)}}{1 - e^{2(\bar{\psi}(X-) - \bar{\psi}(X+))}}.$$

(3.25) follows immediately from (3.26). □

We now give existence theorems for the reduced problem and start with the simple case

(B).

Theorem 3.1. Assume that  $D(x) > D_B > 0$  on  $[-1, 1]$ ,  $D \in C^1([-1, X])$ ,  $D \in C^1([X, 1])$  and that

$$(3.27) \quad S_n(D(x), 0, J, 0, 0) \equiv S_p(D(x), 0, J, 0, 0) \equiv 0 \quad \text{for all } x \in [-1, 1] \\ \text{and all } J \in \mathbb{R}.$$

Then the reduced problem (3.22), (3.19), (3.20), (3.21) has the solution

$$(3.28) \quad \left. \begin{array}{ll} \text{(a)} \quad \bar{n}(x) \equiv D(x) \\ \text{(b)} \quad \bar{p}(x) \equiv 0 \\ \text{(c)} \quad \bar{J}_p(x) \equiv 0 \\ \text{(d)} \quad \bar{J}_n(x) \equiv \frac{U}{U_T} / \int_{-1}^1 \frac{ds}{D(s)} \end{array} \right\} \begin{array}{l} -1 \leq x < X \\ X < x \leq 1 \end{array}$$

$$(e) \quad \bar{\psi}(x) = \begin{cases} \frac{U_A}{U_T} + \ln D(x) - \frac{U}{U_T} \frac{\int_{-1}^x \frac{ds}{D(s)}}{\int_{-1}^1 \frac{ds}{D(s)}}, & -1 \leq x < X \\ \frac{U_C}{U_T} + \ln D(x) - \frac{U}{U_T} \frac{\int_{+1}^x \frac{ds}{D(s)}}{\int_{-1}^1 \frac{ds}{D(s)}}, & X < x \leq 1. \end{cases}$$

Proof: Assume (3.28)(a),(b),(c). Then (3.27), (3.19)(c) imply that  $\bar{J}_n \equiv \text{const}$  on  $[-1, 1]$ . From (3.22)(a) we conclude that



$$\bar{\psi}(x) = \begin{cases} \psi_- + \ln \frac{D(x)}{D(-1)} - \bar{J}_n \int_{-1}^x \frac{ds}{D(s)}, & -1 < x < X \\ \psi_+ + \ln \frac{D(x)}{D(1)} - \bar{J}_n \int_1^x \frac{ds}{D(s)}, & X < x < 1 \end{cases}$$

holds. Now (3.22) and all boundary conditions are fulfilled.  $\bar{J}_n$  has to be calculated from (3.19)(a). ((3.19)(b) is automatically fulfilled.) (3.28)(e) follows then by using (3.1)(c),(d). □

For a recombination rate  $R$  which depends only on  $n, p$ , (3.27) is a direct consequence of the equilibrium condition. Therefore Theorem 3.1 holds for the SRH-recombination term.

Assuming the validity of the asymptotic expansions (3.2) (which will be proven later) the theorem implies that the device is depleted of holes (away from the junction) and that the electron current  $J_n$  is asymptotically proportional to the applied voltage  $U$ . Actually (3.28)(d) is a scaled version of Ohm's law ( $\int_{-1}^1 \frac{ds}{D(s)}$  is the (scaled) resistance of the device).  $n^+n$  and  $nn^+$  junctions are resistors.

Now we turn to the case (A). For simplicity we take the SRH-recombination term.

**Theorem 3.2.** Assume that  $D(x) < 0$  on  $[-1, X]$ ,  $D(x) > 0$  on  $[X, 1]$ ,  $|D(x)| > D_A$  on  $[-1, 1]$  and that  $D \in C^1([-1, X])$ ,  $D \in C^1([X, 1])$ . Let  $S_n, S_p$  be given by (2.14) (SRH). Moreover assume that

$$(3.29) \quad e^{\psi_- - \psi_+} < \rho, \rho \text{ sufficiently small}$$

holds. Then the reduced problem (3.22), (3.19), (3.20), (3.21) has a locally unique solution (in  $C^1([-1, X) \cup (X, 1])^4$ ) which fulfills (3.24) and

$$\begin{aligned}
(3.30) \quad (a) \quad \bar{\psi}(x) &= \begin{cases} \psi_- - \ln \frac{D(x)}{D(-1)} + \bar{\phi}(x), & -1 \leq x < X \\ \psi_+ + \ln \frac{D(x)}{D(1)} + \bar{\phi}(x), & X < x \leq 1 \end{cases} \\
\bar{\phi}(x) &= O(e^{\psi_- - \psi_+}) \text{ on } [-1, 1] \\
(b) \quad \bar{n}(x) &= \begin{cases} O(e^{\psi_- - \psi_+}) > 0, & -1 \leq x < X \\ D(x) + O(e^{\psi_- - \psi_+}), & X < x \leq 1 \end{cases} \\
(c) \quad \bar{J}_n(x) &= O(e^{\psi_- - \psi_+}), \quad -1 \leq x \leq 1 \\
(d) \quad \bar{J}_p(x) &= O(e^{\psi_- - \psi_+}), \quad -1 \leq x \leq 1.
\end{aligned}$$

Of course  $\bar{p}(x)$  is then given by

$$(3.30) \quad (e) \quad \bar{p}(x) = \begin{cases} -D(x) + O(e^{\psi_- - \psi_+}), & -1 \leq x < X \\ O(e^{\psi_- - \psi_+}) > 0, & X < x \leq 1. \end{cases}$$

Proof: We introduce  $\bar{\phi}$  as a new dependent variable (instead of  $\bar{\psi}$ ) and obtain from

(3.22)(a)

$$\begin{aligned}
(3.31) \quad (a) \quad \bar{\phi}' &= \frac{2DD' - (\bar{J}_n + \bar{J}_p)D - 2\bar{n}D'}{D(2\bar{n} - D)}, \quad X < x \leq 1 \\
(b) \quad \bar{\phi}' &= \frac{2\bar{n}D' - D(\bar{J}_n + \bar{J}_p)}{D(2\bar{n} - D)}, \quad -1 \leq x < X.
\end{aligned}$$

Boundary conditions for  $\bar{\phi}$  are

$$(3.32) \quad \bar{\phi}(1) = \bar{\phi}(-1) = 0$$

and the interface conditions (3.19)(a),(b) transform to

$$\begin{aligned}
(3.33) \quad (a) \quad e^{\psi_- - \psi_+} \frac{D(-1)D(1)}{D(X-)D(X+)} e^{\bar{\phi}(X-) - \bar{\phi}(X+)} \bar{n}(X+) - \bar{n}(X-) &= 0 \\
(b) \quad e^{\psi_- - \psi_+} \frac{D(-1)D(1)}{D(X-)D(X+)} e^{\bar{\phi}(X-) - \bar{\phi}(X+)} (\bar{n}(X-) - D(X-)) - (\bar{n}(X+) - D(X+)) &= 0.
\end{aligned}$$

We set  $w = e^{\psi - \psi_+}$  and  $\bar{z}_0 = (\bar{\phi}_0, \bar{n}_0, \bar{J}_{n_0}, \bar{J}_{p_0})$ , where  
 $\bar{\phi}_0 \equiv 0$  on  $[-1, 1]$

$$(3.34) \quad \bar{n}_0(x) = \begin{cases} 0, & -1 \leq x \leq X \\ D(x), & X < x \leq 1 \end{cases}$$

$$\bar{J}_{n_0}(x) \equiv \bar{J}_{p_0}(x) \equiv 0 \text{ on } [-1, 1]$$

and write the problem (3.31), (3.22)(b) - (d), (3.20)(a), (3.21)(a), (3.32), (3.33),

(3.19)(c), (d) in operator form  $F(w, \bar{z}) = 0$  where  $F : [0, \infty) \times (C_X^1([-1, 1]))^4 \rightarrow$

$(C_X^1([-1, 1]))^4 \times \mathbb{R}^8$ .  $C_X^1([-1, 1])$  is the space of functions of which are  $i$ -times

continuously differentiable on  $[-1, X]$  and on  $(X, 1]$  and  $\lim_{x \rightarrow X+} f^{(j)}(x)$  exists for

$0 \leq j \leq i$ . The space is equipped with the norm  $\|f\|_{X,i} = \sum_{j=0}^i \sup_{\substack{x \in [-1, 1] \\ x \neq X}} |f^{(j)}(x)|$ .

Obviously  $F(0, \bar{z}_0) = 0$ .

We investigate the equation  $D_z F(0, \bar{z}_0)y = (f, \alpha)$  (where  $D_z F(0, \bar{z}_0)$  denotes the Fréchet derivative of  $F$  with respect to  $z$  at  $(0, \bar{z}_0)$ ) for  $y = (y_1, y_2, y_3, y_4)$  and obtain

$$(3.35) \quad (a) \quad y' = \begin{bmatrix} 0 & -\frac{2D'}{D^2} & -\frac{1}{D} & -\frac{1}{D} \\ 0 & -\frac{D'}{D} & 0 & -1 \\ 0 & \frac{1}{\beta} & 0 & 0 \\ 0 & -\frac{1}{\beta} & 0 & 0 \end{bmatrix} y + f, \quad X < x \leq 1$$

$$(3.35) \quad (b) \quad y' = \begin{bmatrix} 0 & -\frac{2D'}{D^2} & -\frac{1}{D} & -\frac{1}{D} \\ 0 & -\frac{D'}{D} & 1 & 0 \\ 0 & \frac{1}{\beta} & 0 & 0 \\ 0 & -\frac{1}{\beta} & 0 & 0 \end{bmatrix} y + f, \quad -1 \leq x < X$$

where  $f \in (C_x([-1,1]))^4$ . The boundary and interface conditions are

$$\begin{aligned}
 (3.35) \quad & (c) \quad y_1(-1) = \alpha_1, \quad y_2(-1) = \alpha_2 \\
 & (d) \quad y_1(1) = \alpha_3, \quad y_2(1) = \alpha_4 \\
 & (e) \quad y_3(x+) - y_3(x-) = \alpha_5 \\
 & (f) \quad y_4(x+) - y_4(x-) = \alpha_6 \\
 & (g) \quad y_2(x-) = \alpha_7 \\
 & (h) \quad y_2(x+) = \alpha_8
 \end{aligned}$$

with  $\alpha_i \in \mathbb{R}$ ,  $i=1, \dots, 8$ .

Because of the Fredholm alternative  $D_z F(0, \bar{z}_0)$  is one-to-one and onto iff it is one-to-one. Therefore we only have to show that the homogeneous problem (3.35) ( $\alpha=0$ ,  $f \equiv 0$ ) has the unique solution  $y \equiv 0$ .

From (3.35)(a) we get

$$(3.36) \quad \left. \begin{aligned} y_2' &= -\frac{D'}{D} y_2 - y_4 \\ y_3' &= \frac{1}{\beta} y_2 \\ y_4' &= -\frac{1}{\beta} y_2 \end{aligned} \right\} x < x < 1$$

Therefore

$$y_4'' = -\frac{1}{\beta} y_2' = -\frac{1}{\beta} \left( -\frac{D'}{D} y_2 - y_4 \right) = -\frac{1}{\beta} \left( \beta \frac{D'}{D} y_4 - y_4 \right)$$

and

$$(3.37)(a) \quad y_4'' = (\ln D)' y_4' - \frac{1}{\beta} y_4 = 0, \quad x < x < 1$$

holds. The boundary conditions are

$$(3.37)(b) \quad y_4'(1) = y_4'(x+) = 0.$$

Since  $\beta > 0$  the maximum principle implies that  $y_4 \equiv 0$  on  $[x, 1]$  and therefore

$y_3 \equiv y_2 \equiv y_1 \equiv 0$  on  $[x, 1]$ .

(3.35)(b) gives

$$(3.38) \quad \left. \begin{aligned} y_2' &= -\frac{D'}{D} y_2 + y_3 \\ y_3' &= \frac{1}{\beta} y_2 \\ y_4' &= \frac{1}{\beta} y_2 \end{aligned} \right\} -1 \leq x \leq X$$

Again we get a second order problem

$$(3.39)(a) \quad y_3'' + (\ln |D|)' y_3' - \frac{1}{\beta} y_3 = 0, \quad -1 \leq x \leq X$$

$$(3.39)(b) \quad y_3'(-1) = y_3'(X) = 0$$

and the maximum principle yields  $y_3 \equiv 0$  on  $[-1, X]$ .  $y_1 \equiv y_2 \equiv y_3 \equiv 0$  on  $[-1, X]$  follows immediately.

Therefore  $D_z F(0, \bar{z}_0)$  is an isomorphism and since  $D_z F(w, \bar{z}_0)$  is uniformly Lipschitz continuous the implicit function theorem assures that there is a locally unique solution  $\bar{z} = \bar{z}(w)$  of  $F(w, \bar{z}) = 0$  for  $w \in [0, w_0]$  sufficiently small. Since

$$F(w, \bar{z}_0) = O(w)$$

we get  $\|\bar{z}(w) - \bar{z}_0\|_{X,1} = O(w)$ . To show that this solution  $\bar{z}(w)$  fulfills (3.24) we compute the first order term  $\bar{z}_1$  of the expansion

$$\bar{z}(w) \sim \sum_{i=0}^{\infty} w^i \bar{z}_i$$

as a solution of the equation

$$D_z F(0, \bar{z}_0) \bar{z}_1 = -D_w F(0, \bar{z}_0) \quad .$$

$\bar{z}_1$  solves (3.35)(a),(b) with  $f \equiv 0$ , fulfills the interface conditions (setting  $\bar{z}_1 = (\bar{\psi}_1, \bar{n}_1, \bar{J}_{n_1}, \bar{J}_{p_1})$ )

$$\bar{n}_1(x-) = \frac{D(-1)D(1)}{D(x-)} > 0$$

$$\bar{n}_1(x+) = -\frac{D(-1)D(1)}{D(x+)} > 0$$

$$\bar{J}_{n_1}(x+) - \bar{J}_{n_1}(x-) = 0$$

$$\bar{J}_{p_1}(x+) - \bar{J}_{p_1}(x-) = 0$$

and the boundary conditions

$$\bar{\psi}_1(-1) = \bar{\psi}_1(1) = 0$$

$$\bar{n}_1(-1) = \bar{n}_1(1) = 0.$$

$\bar{n}_1, \bar{J}_{n_1}, \bar{J}_{p_1}$  fulfill (3.36) and (3.38) and therefore  $\bar{J}_{p_1}$  fulfills (3.37)(a) on  $[X, 1]$

subject to the boundary conditions

$$\bar{J}'_{p_1}(1) = 0, \bar{J}'_{p_1}(x+) = \frac{1}{\beta} \frac{D(-1)D(1)}{D(x+)} < 0.$$

The maximum principle implies that  $\bar{J}'_{p_1} = -\frac{1}{\beta} \bar{n}_1$  is negative on  $[X, 1]$ , such that  $\bar{n}_1 > 0$  on  $[X, 1]$  holds.

Similarly we obtain  $\bar{n}_1 > 0$  on  $(-1, X]$ . Since the zeros  $x = +1, -1$  of  $\bar{n}_1$  are simple zeros, we obtain (3.24). □

The biggest restriction of the Theorem 3.2 is the required smallness of  $e^{\psi_- - \psi_+}$ . We obtain from (3.1)(a) and (2.24)

$$(3.40) \quad e^{\psi_- - \psi_+} = e^{\frac{U}{U_T} + \frac{U_{bi}(\lambda)}{U_T}}.$$

Therefore (3.29) holds if there is a constant  $K > 0$  sufficiently large (but independent of  $\lambda$ ) such that

$$(3.41) \quad \frac{U}{U_T} < \frac{|U_{bi}(\lambda)|}{U_T} - K$$

holds. The applied voltage  $U$  has to be sufficiently smaller than the absolute value of the built-in voltage (low-injection condition).



Numerical calculations (given in Section 5) demonstrate that (3.41) is not necessary for the existence of a reduced solution.

From (3.30)(c)(d) we get the reduced voltage-current characteristic

$$(3.42) \quad |\bar{J}_n| + |\bar{J}_p| = 0 \left( \frac{\gamma^4 \lambda^4}{D(1)|D(-1)|} e^{\frac{U}{U_T}} (1 + O(\gamma^4 \lambda^4)) \right).$$

In the case (A) (pn-junction) the total current density depends exponentially on the applied voltage (see also Sze (1969)). (3.42) should be compared to the corresponding result (3.28)(c),(d) for the case (B).

Theorem 3.2 can easily be generalized to more general functions  $S_n, S_p$  which do not depend on  $J_n, J_p$ . (3.30) holds without change for  $S_n \equiv S_p \equiv 0$ .

Now we turn to the internal layer problem (3.23). We prove

**Theorem 3.3: Case A:** Set  $D(X+) > 0, D(X-) < 0$ . Then, if the reduced problem (3.22), (3.19), (3.20), (3.21) has a solution fulfilling (3.24), the internal layer problem (3.23) has a unique piecewise monotone solution  $\hat{\psi}$  which fulfills:

$$(a) \quad \begin{aligned} 0 < \hat{\psi}(\sigma) < C_\delta \exp((1-\delta)\sqrt{\bar{n}(X-) + \bar{p}(X-)} \sigma + D_\delta \sqrt{\bar{\psi}(X+) - \bar{\psi}(X-)}) \\ \text{for } \sigma < -E_\delta \sqrt{\bar{\psi}(X+) - \bar{\psi}(X-)} \end{aligned}$$

$$(b) \quad \begin{aligned} 0 < -\hat{\psi}(\sigma) < C_\delta \exp((-1+\delta)\sqrt{\bar{n}(X+) + \bar{p}(X+)} \sigma + D_\delta \sqrt{\bar{\psi}(X+) - \bar{\psi}(X-)}) \\ \text{for } \sigma > E_\delta \sqrt{\bar{\psi}(X+) - \bar{\psi}(X-)} \end{aligned}$$

for every  $\delta > 0$  where  $C_\delta > 0, D_\delta > 0, E_\delta > 0$  depend on  $\delta$  but not on  $\bar{\psi}(X\pm)$  if  $\bar{\psi}(X+) - \bar{\psi}(X-)$  is sufficiently large.

**Case B:** Let  $D$  fulfill the assumptions of Theorem 3.1 and let the reduced solution be given by (3.28). Then the internal layer problem (3.23) has a unique piecewise monotone solution  $\hat{\psi}$  which fulfills

$$(3.44) \quad \begin{aligned} (a) \quad |\hat{\psi}(\sigma)| &< C_\delta \exp((-1-\delta)\sqrt{D(X-)} \sigma), \quad \sigma < 0 \\ (b) \quad |\hat{\psi}(\sigma)| &< C_\delta \exp((-1+\delta)\sqrt{D(X+)} \sigma), \quad \sigma > 0 \end{aligned}$$

for  $\delta > 0$ , where  $C_\delta$  only depends on  $\delta$ .

Piecewise monotone means monotone on  $(-\infty, 0)$  and on  $(0, \infty)$ .

**Proof:** For any piecewise monotone solution  $\hat{\psi}$  of (3.23)

$$(3.45) \quad (a) \quad \text{sgn } \hat{\psi}(0+) = -\text{sgn } \hat{\psi}(0-), \quad (b) \quad \hat{\psi}(0+) \neq 0, \quad \hat{\psi}(0-) \neq 0$$

has to hold. This follows from the monotonicity and from (3.23)(c),(d),(f) since  $\hat{\psi}(0+) = \hat{\psi}(0-) = 0$  would imply  $\hat{\psi} \equiv 0$  (because  $\bar{n}(X\pm) - \bar{p}(X\pm) = D(X\pm)$  holds) which contradicts (3.23)(e) because of (3.25) and (3.28).  $\hat{\psi}(0+) = 0$  (or  $\hat{\psi}(0-) = 0$ ) contradicts (3.23)(e), too.

Only two possible cases remain:

$$\begin{aligned} (I) \quad & \hat{\psi}(0+) > 0, \quad \hat{\psi}(0-) < 0 \\ (II) \quad & \hat{\psi}(0-) < 0, \quad \hat{\psi}(0+) > 0 \end{aligned}$$

In the case (I)  $\hat{\psi}$  has to be monotonically increasing on  $(-\infty, 0)$  and on  $(0, \infty)$ , in the case (II)  $\hat{\psi}$  is decreasing on both intervals. In the case (I) we derive from Fife (1973, Lemma 2.1) that every piecewise monotone solution of (3.23)(a),(b),(c),(d) fulfills

$$(3.46)(a) \quad \sigma = \int_{\hat{\psi}(\sigma)}^{\hat{\psi}(0+)} \frac{d\tau}{\sqrt{2G(\tau')}} \quad , \quad \sigma > 0$$

where

$$\begin{aligned} (3.46)(b) \quad G(\tau) &= \int_0^\tau (\bar{n}(X+)e^s - \bar{p}(X+)e^{-s} - D(X+))ds \\ &= \bar{n}(X+)(e^\tau - 1) + \bar{p}(X+)(e^{-\tau} - 1) - D(X+)\tau \end{aligned}$$

holds ((2.23)(a) fulfills all assumptions of Lemma 2.1 in Fife (1973) because of (3.24))

and

$$(3.47)(a) \quad \sigma = \int_{\hat{\psi}(0-)}^{\hat{\psi}(\sigma)} \frac{d\tau}{\sqrt{2F(\tau)}} \quad , \quad \sigma < 0$$

with

$$(3.47)(b) \quad F(\tau) = \int_0^\tau (\bar{p}(X-)e^s - \bar{n}(X-)e^{-s} + D(X-))ds \\ = \bar{p}(X-)(e^\tau - 1) + \bar{n}(X-) + D(X-)\tau$$

(also (2.3)(b) fulfills the necessary assumptions after setting  $-\phi = \psi$ ,  $\lambda = -\sigma$ ).

Differentiation of (3.46)(a), (3.47)(a) gives

$$(3.48) \quad \hat{\psi}(0+) = -\sqrt{2G(\hat{\psi}(0+))}, \quad \hat{\psi}(0-) = -\sqrt{2F(-\hat{\psi}(0-))}$$

(3.23)(e), (f) gives the equation for  $\hat{\psi}(0+)$

$$(3.49) \quad G(\hat{\psi}(0+)) = F(\bar{\psi}(X-) - \bar{\psi}(X+) - \hat{\psi}(0+))$$

which can be solved (uniquely) by using (3.46)(b), (3.47)(b) giving

$$(3.50)(a) \quad \hat{\psi}(0+) = \frac{D(X-)(\bar{\psi}(X+) - \bar{\psi}(X-)) + (\bar{n}(X-) - \bar{n}(X+)) + (\bar{p}(X-) - \bar{p}(X+))}{D(X-) - D(X+)}$$

(3.23)(e) implies

$$(3.50)(b) \quad \hat{\psi}(0-) = \frac{D(X+)(\bar{\psi}(X+) - \bar{\psi}(X-)) + (\bar{n}(X-) - \bar{n}(X+)) + (\bar{p}(X-) - \bar{p}(X+))}{D(X+) - D(X-)}$$

In the case (II) we proceed analogously and obtain the same formulas for  $\hat{\psi}(0+)$ ,  $\hat{\psi}(0-)$ .

Therefore, a unique piecewise monotone solution of (3.23) exists iff  $\hat{\psi}(0+)$ ,  $\hat{\psi}(0-)$  as given by (3.50), have appropriate sign (and are not zero).

In the case (B) (3.50), (3.28) give

$$(3.51) \quad (a) \quad \hat{\psi}(0+) = \frac{D(X-) \ln \frac{D(X+)}{D(X-)} + D(X-) - D(X+)}{D(X+) - D(X-)} \\ (b) \quad \hat{\psi}(0-) = \frac{D(X+) \ln \frac{D(X+)}{D(X-)} + D(X-) - D(X+)}{D(X+) - D(X-)}$$

Setting  $y = \frac{D(X+)}{D(X-)} > 0$  we define

$$(3.52) \quad (a) \quad \hat{\psi}(0+) = f_1(y) = \frac{\ln y - (y-1)}{y-1} \\ (b) \quad \hat{\psi}(0-) = f_2(y) = \frac{y \ln y - (y-1)}{y-1}$$

Obviously  $\ln y - (y-1) < 0$  holds for  $y > 0$ ,  $y \neq 1$ . Also  $y \ln y - (y-1) > 0$  for  $y > 0$ ,  $y \neq 1$  holds and we find that  $\text{sgn } f_1(y) = -\text{sgn } f_2(y)$ ,  $f_1(y) \neq 0$ ,  $f_2(y) \neq 0$  for  $y > 0$ ,

$y \neq 1$ . We derive from (3.52)

$$(3.53) \quad \begin{aligned} (a) \quad & \hat{\psi}(0+) > 0, \quad \hat{\psi}(0-) < 0 \quad \text{iff} \quad D(X+) < D(X-) \\ (b) \quad & \hat{\psi}(0+) < 0, \quad \hat{\psi}(0-) > 0 \quad \text{iff} \quad D(X+) > D(X-) . \end{aligned}$$

In the case (A) we express  $\bar{n}(X_{\pm}), \bar{p}(X_{\pm})$  in terms of  $\bar{\psi}(X_{\pm})$  using the interface condition

(3.19)(a), (b) (as in (3.26)) getting

$$(\bar{n}(X-) - \bar{n}(X+)) + (\bar{p}(X-) - \bar{p}(X+)) = \frac{e^{\bar{\psi}(X-) - \bar{\psi}(X+)} - 1}{e^{\bar{\psi}(X-) - \bar{\psi}(X+)} + 1} (D(X-) + D(X+)). \quad \text{We set } z = \bar{\psi}(X-) - \bar{\psi}(X+)$$

and obtain

$$(3.54) \quad \begin{aligned} (a) \quad & \hat{\psi}(0+) = \frac{h_1(z)}{D(X+) - D(X-)}, \quad (b) \quad \hat{\psi}(0-) = \frac{h_2(z)}{D(X+) - D(X-)} \end{aligned}$$

where

$$(3.55)(a) \quad h_1(z) = D(X+)g_2(z) + D(X-)g_1(z)$$

$$(3.55)(b) \quad h_2(z) = D(X-)g_2(z) + D(X+)g_1(z)$$

with

$$(3.56)(a) \quad g_1(z) = e^z - 1 - z(e^z + 1)$$

$$(3.56)(b) \quad g_2(z) = e^z - 1 .$$

We restrict to  $z < 0$  since  $z > 0$  cannot occur because of Lemma 3.1  $g_2 < 0$  for  $z < 0$

and a simple computation shows that  $g_1(z) > 0$  for  $z < 0$ . Since  $D(X+) > 0, D(X-) < 0$

we obtain  $h_1(z) > 0$  and  $h_2(z) < 0$  for  $z < 0$ . Therefore

$$(3.57) \quad (a) \quad \hat{\psi}(0+) < 0, \quad (b) \quad \hat{\psi}(0-) > 0$$

follows in the case (A).

Now the existence theorem is settled in both cases, the decay statements (3.43),

(3.44) still have to be shown.

In the case (B) the equation (3.23)(b) reads

$$\ddot{\psi} = g(\hat{\psi}) := D(X+)(e^{\hat{\psi}} - 1) .$$

$g'(0) = D(X+)$  holds and (3.44)(b) follows from Fife (1973, Lemma 2.1). (3.44)(a) is

derived in the same way.

In the case (A) we have to keep in mind that  $\bar{\psi}(X+) - \bar{\psi}(X-)$  can be large (see Theorem

3.2), which implies that  $\hat{\psi}(0+)$ ,  $\hat{\psi}(0-)$  are large. Therefore we need estimates which are uniform for large  $\bar{\psi}(X+) - \bar{\psi}(X-)$ .

The proof follows the lines of the proof of Lemma 2.1 in Fife (1973). We set  $\hat{\phi} = -\hat{\psi}$  in (2.23)(b), call the (new) right hand side of  $(\hat{\psi})$  and compute  $\sqrt{f'(0)} = \sqrt{n(X+)+p(X+)}$ . It is easy to show that

$$(3.58) \quad f(s) > (\sqrt{f'(0)} - \delta)^2 s, \quad 0 < s < \delta$$

holds for  $\delta > 0$  sufficiently small. Since  $f$  is increasing we get

$$(3.59) \quad F(\tau) = \int_0^\tau f(s) ds > \frac{(\sqrt{f'(0)} - \delta)^2 \delta^2}{2} + f(\delta)(\tau - \delta)$$

for  $\gamma > \delta$ . Therefore

$$(3.60) \quad (\sqrt{f'(0)} - \delta) \int_\delta^{\hat{\psi}(0+)} \frac{d\tau}{\sqrt{2F(\tau)}} < D_\delta \sqrt{|\hat{\psi}(0+)|}$$

holds for  $|\hat{\psi}(0+)|$  sufficiently large, where  $D_\delta > 0$  is independent of  $\hat{\psi}(0+)$ . From Fife (1973, Lemma 2.1) we obtain

$$(3.61) \quad |\hat{\psi}(\sigma)| < \delta \exp(-(1-\delta)\sqrt{f'(0)}\sigma) + D_\delta \sqrt{|\hat{\psi}(0+)|}$$

for  $\sigma > \frac{D_\delta}{\sqrt{f'(0)}} \sqrt{|\hat{\psi}(0+)|}$ . (3.43)(b) follows from (3.50)(a). The proof of (3.43)(a) is

analogous. □

Similar estimates holds for the derivatives of  $\hat{\psi}$ .  $\hat{n}$ ,  $\hat{p}$  have to be computed using (3.15), (3.16). In the case (B)  $\bar{p} \equiv 0$  holds (since  $\bar{p} \equiv 0$ ).

If the interface condition (3.23)(f) is changed to  $\hat{\psi}(0+) - \hat{\psi}(0-) = O(\lambda)$  then the layer solution  $\hat{\psi}$  changes at most by  $O(\lambda \sqrt{\bar{\psi}(X+) - \bar{\psi}(X-)})$  (in the max-norm). This follows by applying the implicit function theorem to the perturbed equation (3.49). This will be needed for the existence proof in Section 4.

The width of the internal layer at  $x = X$  can be computed from Theorem 3.3.

In the case (B) we obtain

$$(3.62) \quad \begin{aligned} (a) \quad d_+(\lambda) &= o\left(\frac{\lambda}{\sqrt{D(X+)}} \left| \ln\left(\frac{\lambda}{\sqrt{D(X+)}}\right) \right| \right), \quad \lambda \rightarrow 0+ \\ (b) \quad d_-(\lambda) &= o\left(\frac{\lambda}{\sqrt{D(X-)}} \left| \ln\left(\frac{\lambda}{\sqrt{D(X-)}}\right) \right| \right), \quad \lambda \rightarrow 0+ \end{aligned}$$

where  $d_+(\lambda)$  ( $d_-(\lambda)$ ) denotes the width of that part of the layer which is right (left) of  $X$ .

For the case (A) we obtain

$$(3.63) \quad \begin{aligned} (a) \quad d_+(\lambda) &= o\left(\frac{\lambda}{\sqrt{\bar{n}(X+)+\bar{p}(X+)}} \left( \sqrt{\bar{\psi}(X+)-\bar{\psi}(X-)} + \left| \ln\left(\frac{\lambda}{\sqrt{\bar{n}(X+)+\bar{p}(X+)}}\right) \right| \right) \right) \\ (b) \quad d_-(\lambda) &= o\left(\frac{\lambda}{\sqrt{\bar{n}(X-)+\bar{p}(X-)}} \left( \sqrt{\bar{\psi}(X+)-\bar{\psi}(X-)} + \left| \ln\left(\frac{\lambda}{\sqrt{\bar{n}(X-)+\bar{p}(X-)}}\right) \right| \right) \right). \end{aligned}$$

If the low injection condition (3.41) holds Theorem 3.2 gives a (physically relevant) solution of the reduced problem and (3.30), (3.63) give

$$(3.64) \quad \begin{aligned} (a) \quad d_+(\lambda) &= o\left(\frac{\lambda}{\sqrt{D(X+)}} \left( \sqrt{\left| \frac{U}{U_T} + \frac{U_{bi}(\lambda)}{U_T} \right|} + \left| \ln \frac{\lambda}{\sqrt{D(X+)}} \right| \right) \right) \\ (b) \quad d_-(\lambda) &= o\left(\frac{\lambda}{\sqrt{|D(X-)|}} \left( \sqrt{\left| \frac{U}{U_T} + \frac{U_{bi}(\lambda)}{U_T} \right|} + \left| \ln \frac{\lambda}{\sqrt{|D(X-)|}} \right| \right) \right). \end{aligned}$$

These asymptotics are uniform as  $U \rightarrow -\infty$ .

## 5. Existence of Solutions

In this section we prove the existence of solutions of the singularly perturbed problem (2.11), (2.12) using the asymptotic expansions (3.2).

At first we derive an a priori estimate on the number of carrier pairs valid for the cases (A) and (B).

**Theorem 4.1.** Every solution of (2.11), (2.12) which satisfies (2.16), (2.17) and for which  $J_n, J_p$  do not change sign in  $[-1, 1]$  fulfills

$$(4.1) \quad U > 0 \iff J_n > 0, J_p > 0, J_n \neq 0, J_p \neq 0 \text{ on } [-1, 1]$$

$$(4.2) \quad U = 0 \iff J_n \equiv J_p \equiv 0$$

$$(4.3) \quad U < 0 \iff J_n < 0, J_p < 0, J_n \neq 0, J_p \neq 0 \text{ on } [-1, 1]$$

$$(4.4) \quad \gamma^4 \lambda^4 e^{-\frac{|U|}{U_T}} < n(x)p(x) \leq \gamma^4 \lambda^4 e^{\frac{|U|}{U_T}}, \quad x \in [-1, 1]$$

The proof is completely analogous to the proof of Theorem 4.1 in Markowich, Ringhofer, Selberherr and Langer (1982) and requires only the equilibrium condition on the scaled recombination rates  $S_n, S_p$ . For  $U = 0$  the current densities  $J_n, J_p$  vanish and the device is in thermal equilibrium. The np-product is constant  $\gamma^4 \lambda^4$  throughout the device.

The estimate (4.4), the equilibrium condition and the continuity of  $S_n, S_p$  imply that  $S_n, S_p$  are small along a solution when  $\frac{|U|}{U_T}$  is small. In particular, for the SRH-recombination rate (given by (2.14))

$$(4.5) \quad |S_n| = |S_p| < \frac{\gamma^2 \lambda^2}{2\beta} (e^{\frac{|U|}{U_T}} - 1)$$

holds along every solution of (2.11), (2.12). Therefore it is intriguing to set

$$(4.6) \quad S_n \equiv S_p \equiv 0$$

for sufficiently small  $\frac{|U|}{U_T}$ .

We now give existence proofs for (2.11), (2.12) in this case.

For the simple case (B) we show

**Theorem 4.2.** Assume that  $D$  fulfills the assumptions of Theorem 3.1 and that (4.6) holds. Then, if  $0 < \lambda < \lambda_0$  and  $|\frac{U}{U_T}| < \rho$  for  $\rho$  sufficiently small but independent of  $\lambda$  holds, the problem (2.11), (2.18), (2.19), (2.26), (2.27) has a solution which fulfills (2.16), (2.17) and

$$(4.7) \quad \psi(x, \lambda) = \bar{\psi}(x) + \hat{\psi}\left(\frac{x-x}{\lambda}\right) + O(\lambda)$$

$$(4.8) \quad n(x, \lambda) = D(x) + \hat{n}\left(\frac{x-x}{\lambda}\right) + O(\lambda)$$

$$(4.9) \quad p(x, \lambda) = O(\lambda) \quad (> 0)$$

$$(4.10) \quad J_n(x, \lambda) = \bar{J}_n(x) + O(\lambda)$$

$$(4.11) \quad J_p(x, \lambda) = O(\lambda)$$

uniformly on  $[-1, 1]$  where  $\bar{\psi}, \bar{J}_n$  are given by (3.28)(e), (d),  $\hat{\psi}$  is as in Theorem 3.3, case B and  $\hat{n}$  fulfills (3.15).

**Proof.** The right hand sides of (4.7) - (4.11) are the sum of the reduced solutions as given in Theorem 3.1, the layer terms as of Theorem 3.3 ( $\bar{p} \equiv 0$  holds since  $\bar{p} \equiv 0$ ) and remainder terms. We denote these remainders by  $E_\psi, E_n, E_p, E_{J_n}, E_{J_p}$ . Inserting into (2.11) (with  $S_n \equiv S_p \equiv 0$ ), using (3.23) and (3.6), (3.7) gives

$$\begin{aligned} (a) \quad & \lambda^2 E_\psi'' = E_n - E_p - \lambda^2 \bar{\psi}'' \\ (b) \quad & E_n' = (\bar{\psi} + \hat{\psi})' E_n + (\bar{n} + \hat{n}) E_\psi' + E_{J_n} + E_n E_\psi' + \phi_1(x, \lambda) \\ (4.12)(c) \quad & E_p' = -(\bar{\psi} + \hat{\psi})' E_p - E_{J_p} - E_p E_\psi' + \phi_2(x, \lambda) \\ (d) \quad & E_{J_n}' = 0 \\ (e) \quad & E_{J_p}' = 0 \end{aligned}$$

where the functions  $\phi_1, \phi_2$  satisfy

$$(4.13) \quad \int_{-1}^1 |(\phi_i(s, \lambda))| ds = O(\lambda), \quad i = 1, 2.$$

Inserting into the boundary conditions (2.18), (2.19), (2.26), (2.27) shows that the boundary values for  $E_\psi, E_n, E_p$  at  $x = \pm 1$  are  $O(\lambda)$ .

We define the operators:



$$(4.13) \quad \begin{aligned} (a) \quad (H_n g)(x) &= \int_{-1}^x \exp(\bar{\psi}(x) + \hat{\psi}(\sigma(x)) - \bar{\psi}(s) - \hat{\psi}(\sigma(s))) g(s) ds \\ (b) \quad (H_p g)(x) &= \int_{-1}^x \exp(-\bar{\psi}(x) - \hat{\psi}(\sigma(x)) + \bar{\psi}(s) + \hat{\psi}(\sigma(s))) g(s) ds \end{aligned}$$

where  $\sigma(x) = \frac{x-X}{\lambda}$  and rewrite (4.12) (b), (c) as integral equations

$$(4.14)(a) \quad \begin{aligned} E_n &= \exp(\bar{\psi}(x) + \hat{\psi}(\sigma(x)) - \bar{\psi}(-1) - \hat{\psi}(\sigma(-1))) E_n(-1) + \\ &\quad (H_n(\bar{n} + \hat{n})E'_\psi)(x) + (H_n E_J)_n(x) + (H_n E_n E'_\psi)(x) \\ &\quad + (H_n \phi_1)(x) \end{aligned}$$

$$(4.14)(b) \quad \begin{aligned} E_p &= \exp(-\bar{\psi}(x) - \hat{\psi}(\sigma(x)) + \bar{\psi}(\sigma(-1))) E_p(-1) - \\ &\quad - (H_p E_J)_p(x) - (H_p E_p E'_\psi)(x) + (H_p \phi_2)(x) . \end{aligned}$$

From (4.12)(c), (d) we get

$$(4.15)(a) \quad E_J_n \equiv \text{const. on } [-1, 1]$$

$$(4.15)(b) \quad E_J_p \equiv \text{const. on } [-1, 1]$$

since  $E_{J_n}, E_{J_p} \in C([-1, 1])$ .

Because of (4.13) and since  $|E_n(-1)| = 0(\lambda)$ ,  $|E_p(-1)| = 0(\lambda)$  we obtain from (4.14)

$$(4.16) \quad \begin{aligned} (a) \quad E_n &= H_n(\bar{n} + \hat{n})E'_\psi + E_J H_n 1 + H_n E_n E'_\psi + 0(\lambda) \\ (b) \quad E_p &= -E_J H_p 1 + H_p E_p E'_\psi + 0(\lambda) . \end{aligned}$$

Partial integration and (3.23) give

$$(4.17) \quad (H_n(\bar{n} + \hat{n})E'_\psi)(x) = (\bar{n} + \hat{n})E_\psi - (H_n \bar{J}_n E_\psi)(x) + \lambda G_{n, \psi} E_\psi$$

where  $G_{n, \psi} : C([-1, 1]) \rightarrow C([-1, 1])$  is uniformly bounded (in  $\lambda$ ). The continuity of  $\bar{n} + \hat{n}$  at  $x = X$  was used for the derivation of (4.17).

From (4.12)(b) we derive, after partial integration

$$(4.18) \quad H_n E_n E'_\psi = E_n E_\psi - H_n \bar{n} E'_\psi E_\psi - H_n E_J E_\psi - H_n E_n E'_\psi E_\psi + \lambda F_{n, \lambda} E'_\psi E_\psi + 0(\lambda)$$

where  $F_{n,\lambda} : C([-1,1]) \rightarrow C([-1,1])$  is uniformly bounded. Another partial integration gives

$$(4.19) \quad H_n \bar{n} E'_\psi E_\psi = \frac{1}{2} \bar{n} E_\psi^2 - e^{\bar{\psi}} \int_{-1}^x (e^{-\bar{\psi}_n})' E_\psi^2(s) ds + O(\lambda) .$$

Combining (4.16) - (4.19) gives

$$(4.20)(a) \quad E_n = (\bar{n} + n) E_\psi - H_n \bar{J}_n E_\psi + E_{J_n} H_n 1 + \Lambda_{n,\lambda}(E_\psi, E'_\psi, E_n, E_{J_n})$$

and proceeding analogously for  $E_p$

$$(4.20)(b) \quad E_p = -E_p H_p 1 + \Lambda_{p,\lambda}(E_\psi, E'_\psi, E_p, E_{J_p})$$

where  $\Lambda_{n,\lambda}, \Lambda_{p,\lambda}$  are nonlinear operators from  $(C([-1,1]))^4$  into  $C([-1,1])$  which fulfill

$$(4.21)(a) \quad \begin{aligned} \|\Lambda_{q,\lambda}(E_\psi, E'_\psi, E_q, E_{J_q})\|_{[-1,1]} &< C_1(0(\lambda) + \|E_\psi\|_{[-1,1]}^{(\lambda+\lambda\|E'_\psi\|_{[-1,1]})} \\ &+ \|E_\psi\|_{[-1,1]} + \|E_{J_q}\|_{[-1,1]} + \|E_q\|_{[-1,1]} \|E'_\psi\|_{[-1,1]}) \end{aligned}$$

$$(4.21)(b) \quad \begin{aligned} \|D\Lambda_{q,\lambda}(E_\psi, E'_\psi, E_q, E_{J_q})\| &< C_2(\lambda + \|E_\psi\|_{[-1,1]} \\ &+ \|E_q\|_{[-1,1]} + \|E_{J_q}\|_{[-1,1]} \\ &+ \|E_q\|_{[-1,1]} \|E'_\psi\|_{[-1,1]} + \|E_q\|_{[-1,1]} \|E_{J_q}\|_{[-1,1]} \\ &+ \|E'_\psi\|_{[-1,1]} \|E_\psi\|_{[-1,1]} + \lambda \|E'_\psi\|_{[-1,1]}) \end{aligned}$$

where  $C_1, C_2$  are independent of  $\lambda$ , "D" denotes the Fréchet derivative and  $q = n, p$ . The constants  $J_n, J_p$  can be determined from the boundary conditions for  $E_n, E_p$  at  $x = +1$ :

$$(4.22)(a) \quad J_n \equiv \frac{(H_n \bar{J}_n E_\psi)(1) - \Lambda_{n,\lambda}(E_\psi, E'_\psi, E_n, E_{J_n})(1)}{(H_n 1)(1)}$$

$$(4.22)(b) \quad J_p \equiv \frac{\Lambda_{p,\lambda}(E_\psi, E'_\psi, E_p, E_{J_p})(1)}{(H_p 1)(1)} .$$

We remark that  $(H_n 1)(1), (H_p 1)(1)$  are bounded away from zero uniformly in  $\lambda$ .

We regard  $E_\psi, E_n, E_p, E_{J_n}, E_{J_p}$  as dwelling the space  $A_X = C^1([-1,1]) \cap C_X^2([-1,1])$  which is equipped with the norm  $\|f\|_\lambda = \|f\|_{[-1,1]} + \lambda^2 \|f''\|_{[-1,X) \cup (X,1]}$ . (4.21)

implies that the mappings defined by the right hand sides of (4.20) are contractions in spheres of radius  $0(\lambda)$  (centered at 0) and therefore (4.20)(a), (b) can be resolved with respect to  $E_n, E_p$  resp.:

$$(4.23)(a) \quad E_n = (\bar{n} + \hat{n})E_\psi - H_n \bar{J}_n E_\psi + (H_n \bar{J}_n E_\psi)(1) \frac{(H_n 1)(x)}{(H_n 1)(1)} + \Omega_{n,\lambda}(E_\psi, E'_\psi)$$

$$(4.23)(b) \quad E_p = \Omega_{p,\lambda}(E_\psi, E'_\psi).$$

The operators  $\Omega_{n,\lambda}, \Omega_{p,\lambda}$  fulfill the estimates (4.21) when  $\|E_n\|_{[-1,1]}, \|E_p\|_{[-1,1]}, \|E_{J_n}\|_{[-1,1]}, \|E_{J_p}\|_{[-1,1]}$  are substituted by the radius of the sphere in which the contraction mapping theorem is applied.

Inserting (4.23) into (4.12)(a) gives

$$(4.24) \quad \lambda^2 E_\psi'' - (\bar{n} + \hat{n})E_\psi = (H_n \bar{J}_n E_\psi)(1) \frac{(H_n 1)(x)}{(H_n 1)(1)} - (H_n \bar{J}_n E_\psi)(x) + \Omega_{n,\lambda}(E_\psi, E'_\psi) - \Omega_{p,\lambda}(E_\psi, E'_\psi) - \lambda^2 \bar{\psi}''', \quad -1 \leq x < X \quad \text{and} \quad X < x \leq 1$$

subject to  $0(\lambda)$  boundary conditions for  $E_\psi$  at  $x = \pm 1$ .

Since  $\bar{n} + \hat{n}$  is positive and continuous on  $[-1, 1]$  the boundary value problem

$$(4.25) \quad \lambda^2 y'' - (\bar{n} + \hat{n})y = f(x), \quad y(-1) = y_-, \quad y(1) = y_+$$

has a unique solution  $y \in A_X$  for all  $y_-, y_+ \in \mathbb{R}, f \in C_X([-1, 1])$  which fulfills

$$(4.26) \quad \|y\|_\lambda \leq \text{const}(\|f\|_{[-1,1]} + |y_-| + |y_+|).$$

Since all estimates so far are uniform for  $|\frac{U}{U_T}| < \rho$  and since

$$(4.27) \quad \|H_n \bar{J}_n E_\psi\|_{[-1,1]} \leq \text{const } \rho \|E_\psi\|_{[-1,1]}$$

holds with const independent of  $\lambda$  and  $\rho \in [-\rho_0, \rho_0]$  (see (3.28)(d)) the contraction mapping theorem with  $\rho$  sufficiently small assures the existence of a locally unique  $E_\psi$  with

$$(4.28) \quad \|E_\psi\|_{[-1,1]} = 0(\lambda).$$

The existence statement of Theorem 4.2 follows immediately. The positivity of  $p = E_p$  follows by investigating the higher order terms of the expansions similar to the proof of Theorem 3.2. □

This proof does not carry over to the case A since then  $H_n, H_p$  are not uniformly bounded (in  $\lambda$ ) anymore. At first we rewrite (2.11) (with  $S_n \equiv S_p \equiv 0$ ) as a second order problem.

From (2.11)(d), (e) we get

$$(4.29) \quad J_n \equiv \text{const}, J_p \equiv \text{const} \quad \text{on } [-1, 1]$$

(2.11)(b), (c) give

$$(4.30) \quad \begin{aligned} (a) \quad n &= n(1)e^{\psi(x)-\psi(1)} + J_n e^{\psi(x)} \int_1^x e^{-\psi(s)} ds \\ (b) \quad p &= p(1)e^{\psi(1)-\psi(x)} - J_p e^{-\psi(x)} \int_1^x e^{\psi(s)} ds. \end{aligned}$$

$J_n, J_p$  have to be determined from the boundary conditions for  $n, p$  at  $x = -1$

$$(4.31)(a) \quad J_n \equiv \frac{\gamma^2 \lambda^2 \left( e^{-\frac{U_C}{U_T}} - e^{-\frac{U_A}{U_T}} \right)}{\int_{-1}^1 e^{-\psi(s)} ds}$$

$$(4.31)(b) \quad J_p \equiv \frac{\gamma^2 \lambda^2 \left( e^{\frac{U_A}{U_T}} - e^{\frac{U_C}{U_T}} \right)}{\int_{-1}^1 e^{\psi(s)} ds}$$

(4.30), (4.31) immediately give  $n > 0, p > 0$ .

Without loss of generality we set  $U_A = -U_C = \frac{U}{2}$  and obtain by inserting (4.30),

(4.31) into (2.11)(a)

$$(4.32) \quad \begin{aligned} \lambda^2 \psi'' &= 2\gamma^2 \lambda^2 \left( \sinh\left(\psi + \frac{U}{2U_T}\right) + \sinh\left(\frac{U}{2U_T}\right) \left( e^{\psi} \frac{\int_1^x e^{-\psi(s)} ds}{\int_{-1}^1 e^{-\psi(s)} ds} \right. \right. \\ &\quad \left. \left. + e^{-\psi} \frac{\int_1^x e^{\psi(s)} ds}{\int_{-1}^1 e^{\psi(s)} ds} \right) \right) - D(x) \end{aligned}$$

for  $x \in [-1, X) \cup (X, 1]$  subject to the boundary conditions (2.20), (2.21) and the interface conditions (2.16).

At first we derive an asymptotic representation of the voltage current characteristic.

**Lemma 4.1.** Let  $S_n \equiv S_p \equiv 0$  hold and assume that  $U$  fulfills the low injection condition (3.41), that  $D$  fulfills the assumptions of Theorem 3.2 (Case (A)) and

$$(4.33)(a) \quad \psi(x, \lambda) = \bar{\psi}(x) + \hat{\psi}\left(\frac{x-X}{\lambda}\right) + \theta(\lambda)$$

where the reduced solution  $\bar{\psi}$  fulfills (3.30)(a) and  $\hat{\psi}$  is given by Theorem 3.3. Then (3.31) implies

$$(4.33)(b) \quad J_n = \frac{\gamma^4 \lambda^4 (e^{\frac{U}{U_T}} - 1)}{\int_{-1}^X |D(s)| ds} (1 + o(|\theta(\lambda)|) + d_+(\lambda) + d_-(\lambda) + \gamma^4 \lambda^4 e^{\frac{U}{U_T}})$$

$$(4.33)(c) \quad J_p = \frac{\gamma^4 \lambda^4 (e^{\frac{U}{U_T}} - 1)}{\int_X^1 D(s) ds} (1 + o(|\theta(\lambda)|) + d_+(\lambda) + d_-(\lambda) + \gamma^4 \lambda^4 e^{\frac{U}{U_T}})$$

**Proof.** We obtain

$$\begin{aligned} e^{\bar{\psi}_-} \int_{-1}^1 e^{-\bar{\psi}(s, \lambda)} ds &= \int_{-1}^X \frac{D(s)}{D(-1)} ds + \int_{-1}^X \frac{D(s)}{D(-1)} e^{-\hat{\psi}(\frac{s-X}{\lambda})} o(|\theta(\lambda)|) ds \\ &+ \int_{-1}^X \frac{D(s)}{D(-1)} (e^{-\hat{\psi}(\frac{s-X}{\lambda})} - 1) ds + e^{\bar{\psi}_- - \bar{\psi}_+} \int_X^1 \frac{D(1)}{D(s)} ds \\ &+ e^{\bar{\psi}_- - \bar{\psi}_+} \int_X^1 \frac{D(1)}{D(s)} (e^{-\hat{\psi}(\frac{s-X}{\lambda})} - 1) ds \\ &+ e^{\bar{\psi}_- - \bar{\psi}_+} \int_X^1 \frac{D(1)}{D(s)} e^{-\hat{\psi}(\frac{s-X}{\lambda})} o(|\theta(\lambda)|) ds . \end{aligned}$$

Using the estimates (3.43) and the layer widths given by (3.64) we derive

$$e^{-\psi_-} \int_{-1}^1 e^{-\psi(s, \lambda)} ds = \int_{-1}^x \frac{D(s)}{D(-1)} ds + o(|\theta(\lambda)|) + o(d_-(\lambda)) \\ + o(\gamma^4 \lambda^4 e^{\frac{|U|}{U_T}}) + o(d_+(\lambda)) .$$

Therefore

$$J_n = \frac{e^{-\psi_-} \gamma^2 \lambda^2 (e^{\frac{U}{2U_T}} - e^{-\frac{U}{2U_T}})}{e^{-\psi_-} \int_{-1}^1 e^{-\psi(s, \lambda)} ds} = \frac{\gamma^4 \lambda^4 (e^{\frac{U}{U_T}} - 1)}{e^{-\psi_-} \int_{-1}^1 e^{-\psi(s, \lambda)} ds (1 + o(\gamma^4 \lambda^4))} \\ = \frac{\gamma^4 \lambda^4 (e^{\frac{U}{U_T}} - 1)}{\int_{-1}^x |D(s)| ds} (1 + o(|\theta(\lambda)|) + d_+(\lambda) + d_-(\lambda) + \gamma^4 \lambda^4 e^{\frac{U}{U_T}})$$

follows. The proof for the asymptotic representation of  $J_p$  is analogous.

If  $D_n = D_p$  holds (which implies that  $J_n, J_p$  have the same scaling factors) and if a solution of (4.32) subject to the boundary conditions  $\psi(-1, \lambda) = \psi_-$ ,  $\psi(1, \lambda) = \psi_+$  exists for which  $\theta(\lambda) \rightarrow 0$  as  $\lambda \rightarrow 0+$ , then the total voltage current characteristic of the pn-function is given by

$$(4.33)(d) \quad J \equiv J_n + J_p \equiv \gamma^4 \lambda^4 \left( \frac{1}{\int_{-1}^x |D(s)| ds} + \frac{1}{\int_x^1 D(s) ds} \right) (e^{\frac{U}{U_T}} - 1) (1 + o(1)) .$$

The same asymptotic form of  $J$  can be found in Sze (1969) (and other standard books on semiconductor physics), however the derivation used there heavily relies on physical arguments.

We now prove an existence Theorem for the case (A) under a slightly sharper assumption on  $\left| \frac{U}{U_T} \right|$  than that used in Theorem 4.2:

**Theorem 4.3.** Let  $D$  fulfill the assumption of Theorem 3.2 and  $D \in C_X^2([-1, 1])$ . Assume that

$$\left| \frac{U}{U_T} \right| < \frac{\rho}{|\ln \gamma^2 \lambda^2|}$$

holds for some  $\rho > 0$  sufficiently small but independent of  $\lambda$ . Then there is a solution  $\psi(x, \lambda)$  of (4.32) subject to (2.20), (2.21), (2.16) and  $\psi$  fulfills

$$(4.34) \quad \psi(x, \lambda) = \bar{\psi}(x) + \hat{\psi}\left(\frac{x-x}{\lambda}\right) + O(d_+(\lambda) + d_-(\lambda) + (\gamma^4 \lambda^4)^\gamma)$$

for some  $\gamma > 0$  where  $\bar{\psi}$  is the reduced solution as given by Theorem 3.2 (with  $S_n \equiv S_p \equiv 0$ ) and  $\hat{\psi}$  is the layer solution given by Theorem 3.3.

Proof. We define

$$\begin{aligned} \psi_+^0 &= \ln \left[ \frac{D(1) + \sqrt{D(1)^2 + 4\gamma^4 \lambda^4}}{2\gamma^2 \lambda^2} \right] \\ \psi_-^0 &= \ln \left[ \frac{2\gamma^2 \lambda^2}{-D(-1) + \sqrt{D(-1)^2 + 4\gamma^4 \lambda^4}} \right]. \end{aligned}$$

Since  $2\gamma^2 \lambda^2 = \frac{D(1)}{\sinh \psi_+^0} = \frac{D(-1)}{\sinh \psi_-^0}$  holds, the problem (4.32), (2.20), (2.21) with  $U = 0$

can be rewritten as

$$(4.35)(a) \quad \lambda^2 \psi_0'' = D(1) \frac{\sinh \psi_0}{\sinh \psi_+^0} - D(x), \quad X < x \leq 1$$

$$(4.35)(b) \quad \lambda^2 \psi_0'' = D(-1) \frac{\sinh \psi_0}{\sinh \psi_-^0} - D(x), \quad -1 \leq x < X$$

$$(4.35)(c) \quad \psi_0(1, \lambda) = \psi_+^0, \quad \psi_0(-1, \lambda) = \psi_-^0$$

$$(4.35)(d) \quad \psi_0 \in C^1([-1, 1])$$

We now regard  $\psi_+^0, \psi_-^0$  as parameters independent of  $\lambda$  (as in Section 3).

Then the reduced solution  $\bar{\psi}_0$  has the asymptotic form given by Theorem 3.2.

$$(4.36) \quad \bar{\psi}_0(x) = \begin{cases} \text{area} \sinh\left(\frac{D(x)}{D(1)} \sinh \psi_+^0\right) = \psi_+^0 + \ln \frac{D(x)}{D(1)} + O(e^{-2\psi_+^0}), & X < x \leq 1 \\ \text{area} \sinh\left(\frac{D(x)}{D(-1)} \sinh \psi_-^0\right) = \psi_-^0 - \ln \frac{D(x)}{D(-1)} + O(e^{-2\psi_-^0}), & -1 \leq x < X \end{cases}$$

and the internal layer solution  $\hat{\psi}_0$  is as in Theorem 3.3 but subject to the changed

interface condition  $\dot{\bar{\psi}}(0+) - \dot{\bar{\psi}}(0-) = \lambda(\bar{\psi}'(X-) - \bar{\psi}'(X+))$ .

In order to investigate stability we substitute  $u_0 = \frac{\psi_0}{\psi_+}$  in (4.32), (2.20), (2.21)

with  $U = 0$  and obtain the problem

$$(4.37)(a) \quad \lambda^2 u_0'' = \frac{D(1)}{\psi_+^0} \frac{\sinh \psi_+^0 u_0}{\sinh \psi_+^0} - \frac{1}{\psi_+^0} D(x) \quad -1 \leq x < X, \quad X < x \leq 1$$

$$(4.37)(b) \quad u_0(-1) = \frac{\psi_-^0}{\psi_+^0}, \quad u_0(1) = 1$$

$$(4.37)(c) \quad u_0 \in C^1([-1, 1])$$

We denote (4.37) by  $F_0(u_0, \lambda, \psi_+^0, \psi_-^0)$ , where  $F_0(\cdot, \lambda, \psi_+^0, \psi_-^0) : A_X \rightarrow C_X([-1, 1]) \times \mathbb{R}^2$  and investigate the equation

$$(4.38) \quad L_{\lambda, 0} v \equiv D_u F_0(\bar{u}_0 + \hat{u}_0, \lambda, \psi_+^0, \psi_-^0) v = (f, \alpha, \beta)$$

where  $\bar{u}_0 = \frac{\bar{\psi}_0}{\psi_+^0}$ ,  $\hat{u}_0 = \frac{\hat{\psi}_0}{\psi_+^0}$ . (4.38) is equivalent to

$$(4.39)(a) \quad \lambda^2 v'' - D(1) \frac{\sinh(\bar{\psi}_0 + \hat{\psi}_0)}{\sinh \psi_+^0} v = f(x), \quad -1 \leq x < X, \quad X < x \leq 1$$

$$(4.39)(b) \quad v(-1) = \alpha, \quad v(1) = \beta$$

$$(4.39)(c) \quad v \in C^1([-1, 1])$$

We remark that  $\bar{\psi}_0 + \hat{\psi}_0 \in C^1([-1, 1])$ .

The maximum principle immediately implies uniqueness of the solution of (4.39) and the Fredholm's alternative gives existence. To get a bound for the inverse of (4.39) we construct the barrier function:

$$(4.40) \quad v_b(x, \lambda) = K + \exp\left(-\frac{\beta(x-X)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}\right)$$

where the constants  $K > 0$ ,  $\beta > 0$  will be determined thereafter.



We compute

$$(4.41) \quad (L_{\lambda,0} v_b)(x) = \left( \frac{2\beta}{\psi_+^0 - \psi_-^0} \left( \frac{2\beta(x-x)^2}{\lambda^2(\psi_+^0 - \psi_-^0)} - 1 \right) \exp\left(-\frac{\beta(x-x)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}\right) - \right. \\ \left. D(1) \frac{\cosh(\bar{\psi}_0 + \hat{\psi}_0)}{\sinh \psi_+^0} (K + \exp(-\frac{\beta(x-x)^2}{\lambda^2(\psi_+^0 - \psi_-^0)})) \right) , \\ K + \exp(-\frac{\beta(1+x)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}), K + \exp(-\frac{\beta(1-x)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}) .$$

We denote the first component of  $(L_{\lambda,0} v_b)(x)$  by  $(L_{\lambda,0}^{(1)} v_b)(x)$  and get

$$(L_{\lambda,0}^{(1)} v_b)(x) < -\frac{\beta}{\psi_+^0 - \psi_-^0} e^{-\frac{1}{2}} , \quad x \in \left[ x - \frac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}} , x + \frac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}} \right]$$

and

$$\left( \frac{2\beta}{\psi_+^0 - \psi_-^0} \left( \frac{2\beta(x-x)^2}{\lambda^2(\psi_+^0 - \psi_-^0)} - 1 \right) \exp\left(-\frac{\beta(x-x)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}\right) \right)^+ < \frac{4\beta e^{\frac{3}{2}}}{\psi_+^0 - \psi_-^0} , \quad x \in [-1,1]$$

where  $(f)^+$  denotes the positive part of the function  $f$ .

Now we use the estimates for  $\hat{\psi}$  given in Theorem 3.3 for fixed  $0 < \delta < 1$ . At first we estimate

$$D(1) \frac{\cosh(\bar{\psi}_0 + \hat{\psi}_0)}{\sinh \psi_+^0} > \begin{cases} D(x) e^{\hat{\psi}_0} , & x \in [X, 1] \\ |D(x)| e^{-\hat{\psi}_0} , & x \in [-1, X] . \end{cases}$$

Since  $\hat{\psi}_0$  is monotonically increasing on  $[-1, X]$  and on  $(X, 1]$  this yields

$$(L_{\lambda,0}^{(1)})(x) \leq \frac{4\beta e^{\frac{3}{2}}}{\psi_+^0 - \psi_-^0} - K \begin{cases} D(x) \exp(\hat{\psi}_0(\frac{\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}})), & x \in \left[ x + \frac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}}, 1 \right] \\ |D(x)| \exp(-\hat{\psi}_0(\frac{\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}})), & x \in \left[ -1, x - \frac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}} \right] \end{cases}$$

(note that  $\sigma = \frac{x-X}{\lambda}$ ). We now choose  $\beta$

$$\sqrt{\beta} = \frac{1}{2} \min\left(\frac{1}{2E_\delta}, \frac{(1-\delta)\sqrt{\bar{n}(X+) + \bar{p}(X+)}}{2D_\delta}, \frac{(1-\delta)\sqrt{\bar{n}(X-) + \bar{p}(X-)}}{2D_\delta}\right)$$

such that

$$(L_{\lambda,0}^{(1)} v_b)(x) \leq \frac{4\beta e^{\frac{3}{2}}}{\psi_+^0 - \psi_-^0} - K e^{-1} \begin{cases} D(x), & x \in \left[ x + \frac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}}, 1 \right] \\ |D(x)|, & x \in \left[ -1, x - \frac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}} \right] \end{cases}$$

holds and

$$K = \frac{1 + \frac{4\beta e^{\frac{3}{2}}}{\psi_+^0 - \psi_-^0}}{e^{-1} \min_{x \in [-1,1]} |D(x)|}.$$

Then

$$(L_{\lambda,0}^{(1)} v_b)(x) \leq -\min\left(1, \frac{\beta e^{-\frac{1}{2}}}{\psi_+^0 - \psi_-^0}\right), \quad x \in [-1,1]$$

holds. The maximum principle implies the estimate

$$\|L_{\lambda,0}^{-1}\|_{C_X([-1,1]) \times \mathbb{R}^2 \rightarrow C([-1,1])} \leq \text{const.} (\psi_+^0 - \psi_-^0) .$$

(4.39)(a) gives

$$(4.42) \quad \|L_{\lambda,0}^{-1}\|_{C_X([-1,1]) \times \mathbb{R}^2 \rightarrow A_X} \leq \text{const.} (\psi_+^0 - \psi_-^0) .$$

The Fréchet-derivative  $D_u F_0$  is locally Lipschitz continuous:

$$(4.43) \quad \|D_u F_0(w_1, \lambda, \psi_+^0, \psi_-^0) - D_u F_0(w_2, \lambda, \psi_+^0, \psi_-^0)\|_{A_X \times C_X([-1,1]) \times \mathbb{R}^2} \leq \text{const.} \psi_+^0 \|w_1 - w_2\|_{A_X}$$

for  $w_1, w_2$  in a sphere centered at  $\bar{u}_0 + \hat{u}_0 = \frac{\bar{\psi}_0 + \psi_0^0}{\psi_+^0}$  with radius  $\frac{\text{const.}}{\psi_+^0}$ .

Now we rewrite (4.32) as

$$(4.44) \quad \lambda^2 u'' = \frac{D(1)}{\psi_+^0} \frac{\sinh(\psi_+^0 u + \frac{U}{2U_T})}{\sinh \psi_+^0} + \frac{D(1)}{\psi_+^0} \sinh(\frac{U}{2U_T}) \frac{H(\psi_+^0 u)}{\sinh \psi_+^0} - \frac{D(x)}{\psi_+^0}, \quad x \in [-1, X) \cup (X, 1]$$

where  $H(\psi)$  denotes the integral operator on the right hand side of (4.32). We denote

(4.44) subject to the boundary conditions  $u(-1) = \frac{\psi_-}{\psi_+}$ ,  $u(1) = \frac{\psi_+}{\psi_+}$  (with  $\psi_- = \psi_-^0 + \frac{U}{2}$ ,  $\psi_+ = \psi_+^0 - \frac{U}{2}$ ) by  $F_U(u, \lambda, \psi_+^0, \psi_-^0) = 0$ . The Fréchet derivative  $L_{\lambda,U} = D_u F_U(\bar{\psi}_0 + \hat{\psi}_0, \lambda, \psi_+^0, \psi_-^0)$  is given by

$$(4.45) \quad (L_{\lambda,U} v)(x) = (\lambda^2 v'' - \frac{D(1) \sinh(\bar{\psi}_0 + \hat{\psi}_0 + \frac{U}{2U_T})}{\sinh \psi_+^0} - \frac{\sinh \frac{U}{2U_T}}{\sinh \psi_+^0} (D_u H(\bar{\psi}_0 + \hat{\psi}_0) v)(x), \quad v(-1), v(1)) .$$

where

$$\|D_u H(\bar{\psi}_0 + \hat{\psi}_0)\|_{C_X([-1,1]) \rightarrow C_X([-1,1])} \leq \text{const.} \sinh \psi_+^0.$$

Therefore, if

$$(4.46) \quad \left| \frac{U}{U_T} \right| < \frac{\rho}{\psi_+^0 - \psi_-^0}, \quad \rho \text{ suff. small}$$

holds,  $L_{\lambda, U}^{-1}$  fulfills the estimate (4.42). Moreover  $D_u F_U$  is locally Lipschitz continuous and fulfills (4.43).

Since  $\|\bar{\psi} + \hat{\psi} - (\bar{\psi}_0 + \hat{\psi}_0)\|_{[-1,1]} \leq \text{const } |U|$  a simple perturbation argument shows that also  $D_u F_U(\frac{\bar{\psi} + \hat{\psi}}{\psi_+^0}, \lambda, \psi_+^0, \psi_-^0)^{-1}$  fulfills the stability estimate (4.42) and that  $D_u F_U$  is locally

Lipschitz continuous around  $\bar{\psi} + \hat{\psi}$  ((4.43) holds) if  $U$  is restricted by (4.46).

We now insert  $\bar{u} + \hat{u} = \frac{\bar{\psi} + \hat{\psi}}{\psi_+^0}$  into  $F_u$  in order to compute the 'local error'. We obtain

$$F_U(\bar{u} + \hat{u}, \lambda, \psi_+^0, \psi_-^0) = \left( \frac{\lambda^2}{\psi_+^0} \bar{\psi} + \frac{1}{\psi_+^0} \hat{\psi}(\sigma) - \frac{D(1)}{\psi_+^0} \frac{\sinh(\bar{\psi} + \hat{\psi} \frac{U}{2U_T})}{\sinh \psi_+^0} + \right. \\ \left. \frac{D(1)}{\psi_+^0} \sinh\left(\frac{U}{2U_T}\right) \frac{H(\bar{\psi} + \hat{\psi})}{\sinh \psi_+^0} - \frac{D(x)}{\psi_+^0} \right), \frac{\hat{\psi}(\frac{-1-x}{\lambda})}{\psi_+^0}, \frac{\hat{\psi}(\frac{1-x}{\lambda})}{\psi_+^0},$$

with  $\sigma = \frac{x-X}{\lambda}$ . Obviously

$$\frac{D(1)}{\psi_+^0} \sinh\left(\frac{U}{2U_T}\right) \frac{H(\bar{\psi} + \hat{\psi})}{\sinh \psi_+^0} = \frac{1}{\psi_+^0} [\tilde{J}_n e^{\bar{\psi} + \hat{\psi}} \int_1^x e^{-\bar{\psi} - \hat{\psi}} ds + \tilde{J}_p e^{-\bar{\psi} - \hat{\psi}} \int_1^x e^{\bar{\psi} + \hat{\psi}} ds]$$

where  $\tilde{J}_n, \tilde{J}_p$  are given by (4.31)(a), (b) when  $\psi$  is substituted by  $\bar{\psi} + \hat{\psi}$ .

We rewrite

$$\tilde{J}_p e^{-\bar{\psi} - \hat{\psi}} \int_1^x e^{\bar{\psi} + \hat{\psi}} ds = \tilde{J}_p e^{-\bar{\psi} - \hat{\psi}} \int_{-1}^x e^{\bar{\psi} + \hat{\psi}} - \frac{\sinh(\frac{U}{2U_T})}{\sinh \psi_+^0} e^{-\bar{\psi} - \hat{\psi}}.$$

Since  $\hat{\psi}$  is positive on  $[-1, X)$  and negative on  $[X, 1]$  and monotonically decreasing on  $[-1, X)$  and on  $(X, 1]$  we get

$$\left. \begin{aligned} |e^{\bar{\psi}+\hat{\psi}} \int_1^x e^{-\bar{\psi}-\hat{\psi}} ds| &\leq \text{const} \\ |e^{-\bar{\psi}-\hat{\psi}} \int_{-1}^x e^{\bar{\psi}+\hat{\psi}} ds| &\leq \text{const} \end{aligned} \right\} x \in [-1, 1]$$

Lemma (4.1) with  $\theta(\lambda) = 0$  gives

$$\frac{D(1)}{\psi_+^0} \sinh\left(\frac{U}{2U_T}\right) \frac{H(\bar{\psi}+\hat{\psi})}{\sinh \psi_+^0} = 0\left(\gamma^4 \lambda^4 \left|e^{\frac{U}{U_T}} - 1\right|\right) - \frac{D(1) \sinh\left(\frac{U}{2U_T}\right)}{\sinh \psi_+^0} e^{-\bar{\psi}-\hat{\psi}}$$

if the low injection condition holds.

We calculate, using  $\frac{D(1)}{2 \sinh \psi_+^0} = \frac{D(-1)}{2 \sinh \psi_-^0}$ :

$$\begin{aligned} \ddot{\psi}(\sigma) &= [D(1) \frac{\sinh(\bar{\psi}+\hat{\psi} + \frac{U}{2U_T})}{\sinh \psi_+^0} - D(1) \frac{\sinh(\frac{U}{2U_T})}{\sinh \psi_+^0} e^{-\bar{\psi}-\hat{\psi}} - D(x)] \\ &= \ddot{\psi}(\sigma) - [D(1) \frac{e^{\bar{\psi}+\hat{\psi} + \frac{U}{2U_T}}}{2 \sinh \psi_+^0} - D(1) \frac{e^{-\bar{\psi}-\hat{\psi} + \frac{U}{2U_T}}}{2 \sinh \psi_+^0} - D(x)] \\ (4.47) \quad &= \begin{cases} [D(x) \frac{e^{\psi_+^0}}{2 \sinh \psi_+^0} e^{\hat{\psi}} - \frac{D(1)^2 e^{-\psi_+^0 + \frac{U}{U_T}}}{2D(x) \sinh \psi_+^0} e^{-\hat{\psi}} + 0(\gamma^4 \lambda^4 e^{\frac{U}{U_T}}) e^{\hat{\psi}} \\ \frac{D(-1)^2 e^{\psi_-^0 + \frac{U}{U_T}}}{2D(x) \sinh \psi_-^0} e^{\hat{\psi}} - D(x) \frac{e^{-\psi_-^0}}{2 \sinh \psi_-^0} e^{-\hat{\psi}} + 0(\gamma^4 \lambda^4 e^{\frac{U}{U_T}}) e^{\hat{\psi}} \end{cases} \\ &\quad + 0(\gamma^4 \lambda^4 e^{\frac{U}{U_T}}) e^{-\hat{\psi}} - D(x)] , \quad x \in (x, 1] \\ &\quad + 0(\gamma^4 \lambda^4 e^{\frac{U}{U_T}}) e^{-\hat{\psi}} - D(x)] , \quad x \in [-1, x) . \end{aligned}$$

Using Theorem 3.2 gives

$$\left. \begin{aligned}
D(x) \frac{e^{\frac{\psi_+^0}{2 \sinh \psi_+}}}{2 \sinh \psi_+} &= \bar{n}(X+) + (D(x) - D(X+)) + O(\gamma^4 \lambda^4 (1 + e^{\frac{U}{U_T}})) \\
\frac{D(1)^2 e^{-\frac{\psi_+^0 + \frac{U}{U_T}}{2 \sinh \psi_+}}}{2 D(x) \sinh \psi_+} &= \bar{p}(X+) + O(\gamma^4 \lambda^4 (1 + e^{\frac{U}{U_T}}))
\end{aligned} \right\} x \in (X, 1]$$

$$\left. \begin{aligned}
\frac{D(-1)^2 e^{\frac{\psi_-^0 + \frac{U}{U_T}}{2 \sinh \psi_-}}}{2 D(x) \sinh \psi_-} &= \bar{n}(X-) + O(\gamma^4 \lambda^4 (1 + e^{\frac{U}{U_T}})) \\
\frac{D(x) e^{-\frac{\psi_-^0}{2 \sinh \psi_-}}}{2 \sinh \psi_-} &= \bar{p}(X-) + (D(X-) - D(x)) + O(\gamma^4 \lambda^4 (1 + e^{\frac{U}{U_T}}))
\end{aligned} \right\} x \in [-1, X)$$

Since  $\hat{\psi}$  solves the layer-equations (3.23)(a),(b) we get for (4.47)

$$(4.48) \quad \left\{ \begin{aligned}
&(D(x)-D(X+))(e^{\hat{\psi}-1}) + O(\gamma^4 \lambda^4 (1 + e^{\frac{U}{U_T}}))e^{\hat{\psi}} + O(\gamma^4 \lambda^4 (1 + e^{\frac{U}{U_T}}))e^{-\hat{\psi}}, \quad x \in (X, 1] \\
&(D(x)-D(X-))(e^{-\hat{\psi}-1}) + O(\gamma^4 \lambda^4 (1 + e^{\frac{U}{U_T}}))e^{\hat{\psi}} \\
&\quad + O(\gamma^4 \lambda^4 (1 + e^{\frac{U}{U_T}}))e^{-\hat{\psi}}, \quad x \in [-1, X)
\end{aligned} \right.$$

$\hat{\psi}$  is negative on  $(X, 1]$  and positive on  $[-1, X)$  and since  $D \in C_X^1([-1, 1])$  we obtain

$$|(D(x)-D(X+))(e^{\hat{\psi}-1})| \leq \text{const. } d_+(\lambda), \quad x \in (X, 1]$$

$$|(D(x)-D(X-))(e^{-\hat{\psi}-1})| \leq \text{const. } d_-(\lambda), \quad x \in [-1, X)$$

Also, (3.50)(a), (b) imply

$$e^{-\hat{\psi}(\frac{x-x}{\lambda})} \leq e^{-\hat{\psi}(0+)} \leq \text{const.} \exp\left(\frac{|D(x-)|}{D(x+)-D(x-)} (\psi_+ - \psi_-)\right) \\ \leq \text{const.} \exp\left(\frac{|D(x-)|}{D(x+)-D(x-)} \frac{U}{U_T}\right) (\gamma^4 \lambda^4)^{\frac{-|D(x-)|}{D(x+)-D(x-)}}, x \in (x, 1]$$

and analogously

$$e^{\hat{\psi}(\frac{x-x}{\lambda})} \leq e^{\hat{\psi}(0-)} \leq \text{const.} \exp\left(\frac{-D(x+)}{D(x+)-D(x-)} \frac{U}{U_T}\right) (\gamma^4 \lambda^4)^{\frac{-D(x+)}{D(x+)-D(x-)}}, x \in [-1, x)$$

Therefore the expansion (4.48) is bounded by  $\text{const} (d_-(\lambda) + d_+(\lambda) + \exp(\alpha |\frac{U}{U_T}|) (\gamma^4 \lambda^4)^\gamma)$  where  $\alpha > 0, \gamma > 0$  holds.

These estimates and  $\bar{\psi} \in C_X^2([-1, 1])$  for  $D \in C_X^2([-1, 1])$  imply that

$$(4.49) \quad \|F_U(\bar{u} + u, \lambda, \psi_+^0, \psi_-^0)\|_{C_X([-1, 1])} \leq \text{const} \frac{d_-(\lambda) + d_+(\lambda) + e^{\alpha |\frac{U}{U_T}|} (\gamma^4 \lambda^4)^\gamma}{\psi_+^0}$$

holds.

The stability estimate (4.46) (which holds for  $L_{\lambda, U}^{-1}$  if  $U$  fulfills (4.46)), Lipschitz continuity of  $D_U F_U$  and (4.49) make it possible to apply the version of the implicit function theorem given by Spijker (1972), which implies that  $F_U(u, \lambda, \psi_+^0, \psi_-^0) = 0$  has a solution  $u^*$  which is unique in a sphere in  $A_X$  with radius  $\frac{x}{(\psi_+^0)^2}$  for sufficiently small  $x$  centered at  $\bar{u} + u$  and the estimate

$$\|u - u^*\|_{A_X} \leq \text{const.} (d_+(\lambda) + d_-(\lambda) + (\gamma^4 \lambda^4)^\beta)$$

holds for  $|\frac{U}{U_T}| \leq \frac{\rho}{\psi_+^0}$  where  $\rho$  is sufficiently small but independent of  $\lambda$ .

□

We remark that the size reduction on  $|\frac{U}{U_T}|$  comes from the interpretation of (4.32) as perturbation of the equilibrium problem ( $U=0$ ) which was heavily used for the stability proof. The numerical results demonstrated in the next Section indicate that existence and validity of the asymptotic expansions hold under much weaker restrictions on  $|\frac{U}{U_T}|$ .

## 5. Numerical Experiments

We demonstrate numerical results for two pn-junctions in the high injection case, that is  $U \geq U_{bi}(\lambda)$ . The existence Theorem 3.2 for the reduced problem does not hold if the low injection condition (3.41) is neglected. Both functions we investigate have characteristic length  $2l = 5 \times 10^{-3}$  cm, the doping profile of the first pn-junction (called junction I in the sequel) is

$$(5.1) \quad C(z) = \begin{cases} -0.5 \times 10^{17} \text{ cm}^3, & -l < z < \frac{l}{2} \\ 10^{17} \text{ cm}^3, & \frac{l}{2} < z < l \end{cases}$$

and for the second junction (called junction II in the sequel)

$$(5.2) \quad C(z) = \begin{cases} -10^{15} \text{ cm}^3, & -l < z < \frac{l}{2} \\ 10^{17} \text{ cm}^3, & \frac{l}{2} < z < l \end{cases}$$

Accurately speaking, both devices are  $pn^+$  junctions. In both cases  $\bar{C} = 10^{17} \text{ cm}^3$ . This and the numerical values for the parameters from Table 1 gives for both devices using the formulae (2.6), (2.9):

$$(5.3) \quad \lambda^2 = 0.4 \times 10^{-6}, \quad \gamma^2 = 0.25.$$

For junction I we obtain

$$(5.4) \quad \frac{\min|C(z)|}{\frac{z}{\max|C(z)|}} = \frac{1}{2}$$

and for junction II

$$(5.5) \quad \frac{\min|C(z)|}{\frac{z}{\max|C(z)|}} = 10^{-2}.$$

For both cases the singular perturbation approach seems applicable because (2.22), (2.23) 'holds' (the order of magnitude of  $\lambda^2$  and  $\min|C(z)|/\max|C(z)|$  as given by (5.3) and (5.4), (5.5) resp. are clearly different and  $n_i = 10^{10} \text{ cm}^{-3}$  while  $\bar{C} = 10^{17}$ ).



The built-in-voltage (calculated using (2.25)) for junction I is

$$(5.6) \quad U_{bi} = -0.79V$$

and for junction II

$$(5.7) \quad U_{bi} = -0.69V .$$

All calculations described in the sequel were performed on the CDC-Cyber 74 computer of the Technical University of Vienna with the boundary-and-interface-problem solver PASVA4 written by M. Lentini and V. Pereyra. The SRH-recombination term was used. Figures 1-3 show the reduced solutions of a typical high injection case for junction I with  $U = 1.39V$ . The majority carrier densities ( $n$  on the n-side, which is the interval  $(\frac{1}{2}, 1]$  and  $p$  on the p-side which is the interval  $[-1, \frac{1}{2})$ ) are larger than the doping  $|D|$  (except at the boundaries  $x = \pm 1$ ).

The reduced solutions for a high-injection case ( $U = 0.99V$ ) for junction II are shown in Figures 4-6.

Since in both cases the applied voltages are significantly larger than the absolute value of the built-in-voltage, the existence Theorem 3.2 for the reduced problem cannot be applied. However the presented numerical results give a strong indication for the existence of reduced solutions even in the high injection case.

Figure 7-9 and 10-12 show the solutions of the singularly perturbed (full) problem with  $U = |U_{bi}|$  for junction I and II respectively. The internal layer in the components  $\psi$ ,  $n$ ,  $p$  is clearly visible. The solutions of the corresponding reduced problems (whose existence is also not covered by Theorem 3.2) were also computed and they agreed up to graphical accuracy with the full solutions away from the layer (see Figure 13, which shows the reduced solutions  $n$ ,  $p$  for function I). In fact, the reduced solutions were used as starting guesses for the numerical method to compute the full solutions and convergence was achieved in a few steps.

This indicates that the asymptotic expansions are valid for a much larger range of  $U$  values than given in Theorem 4.2.

Figure 1

REDUCED SOLUTION  $D, N, P$   $U = 1.39$

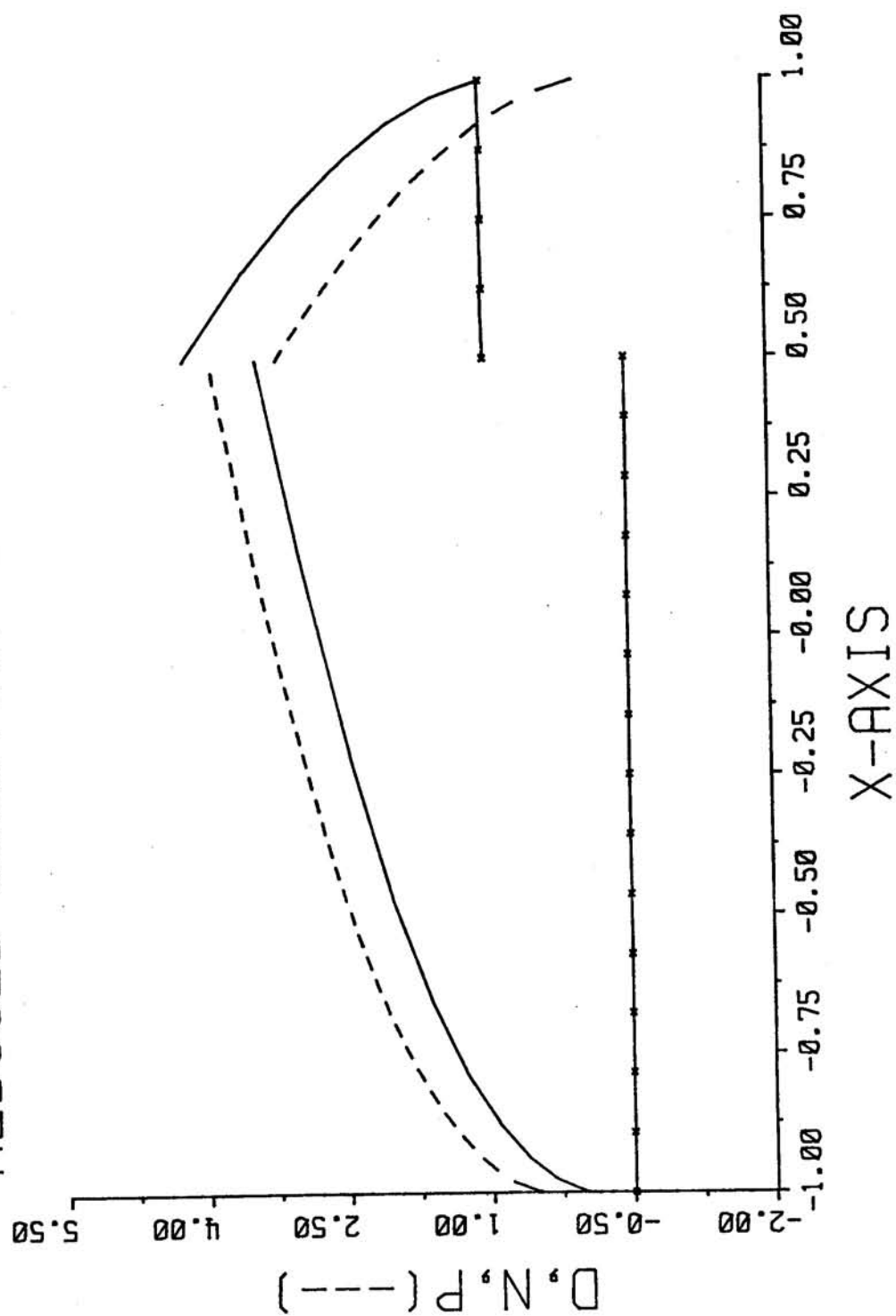


Figure 2

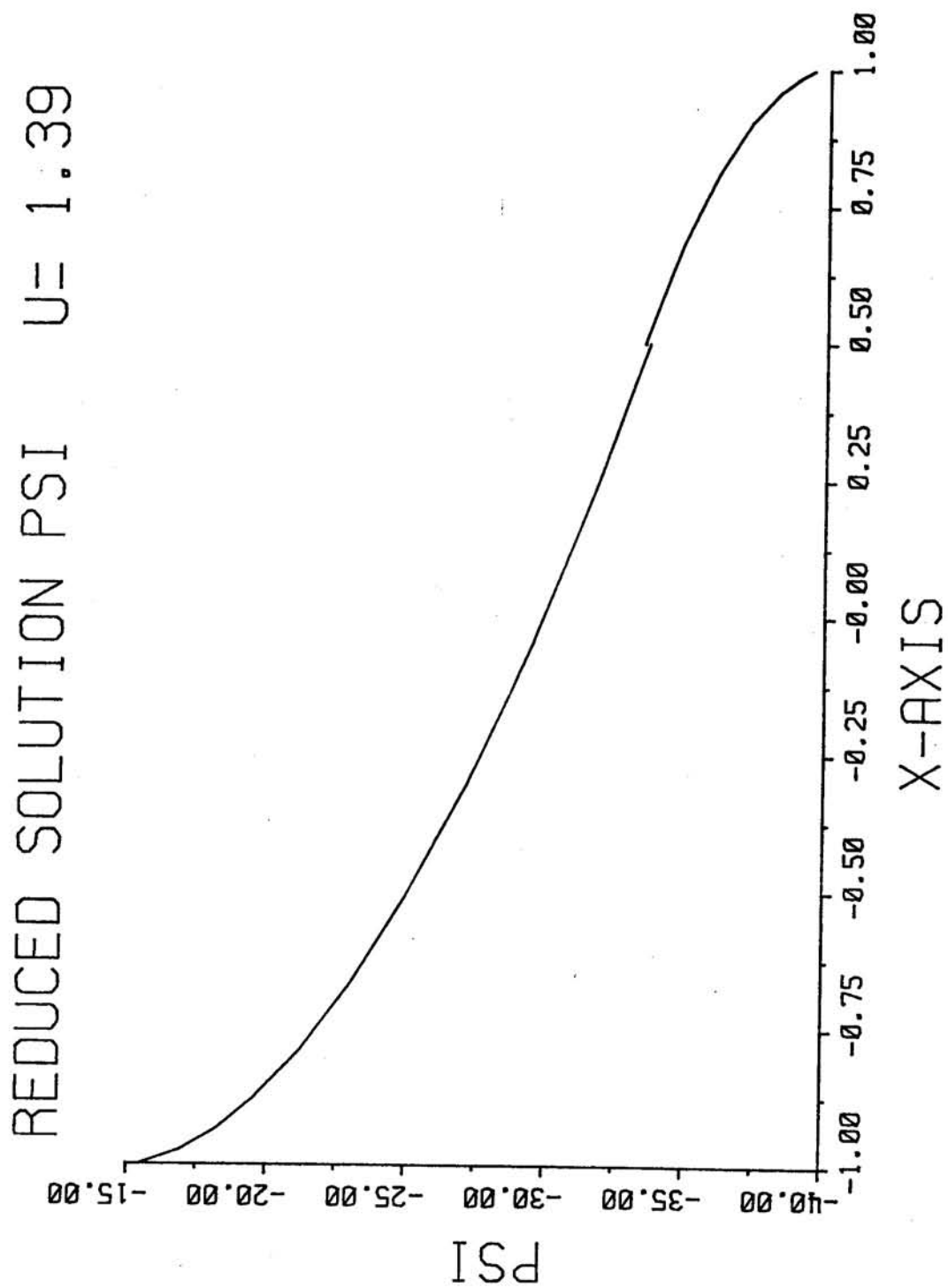


Figure 3

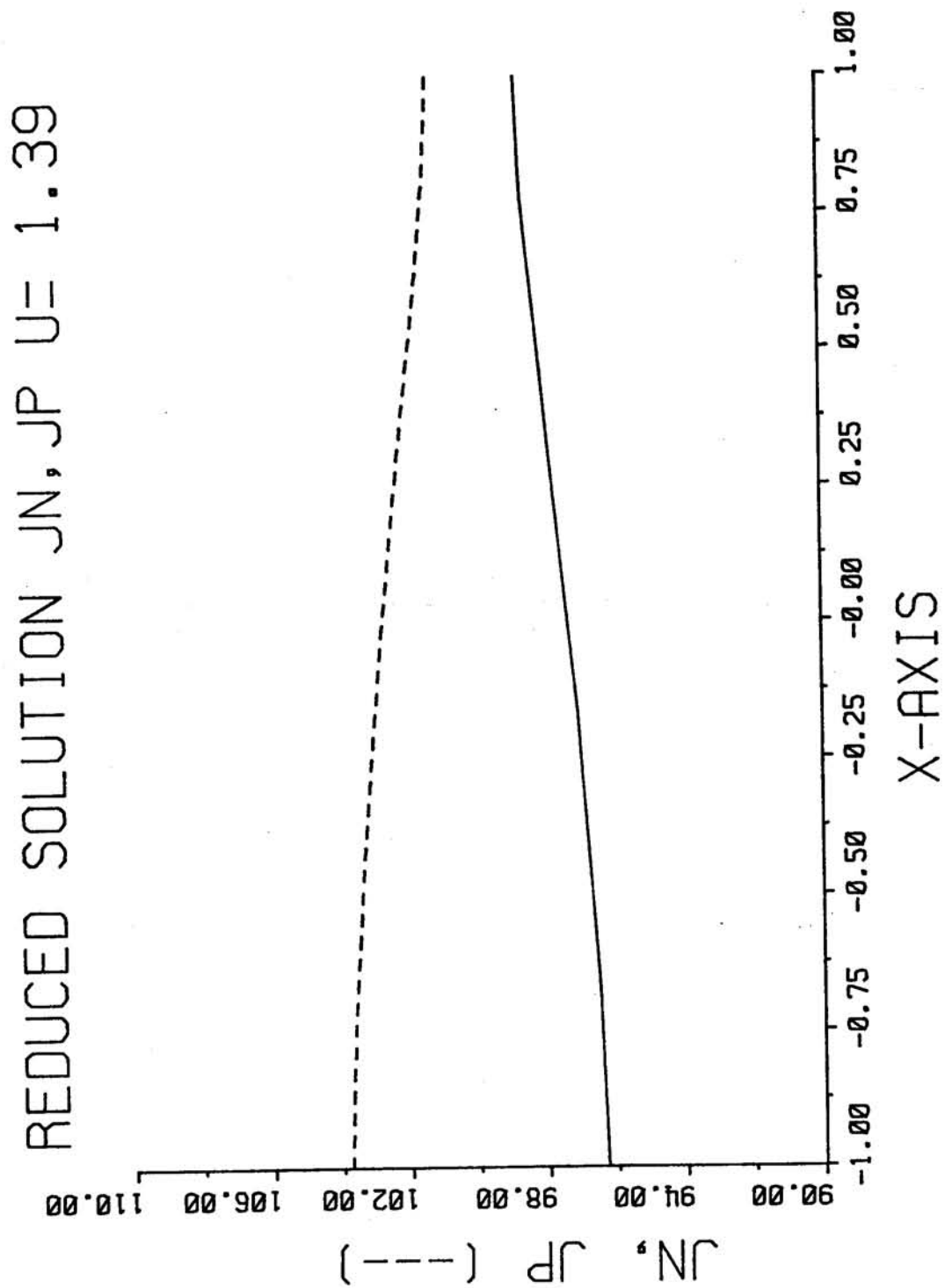


Figure 4

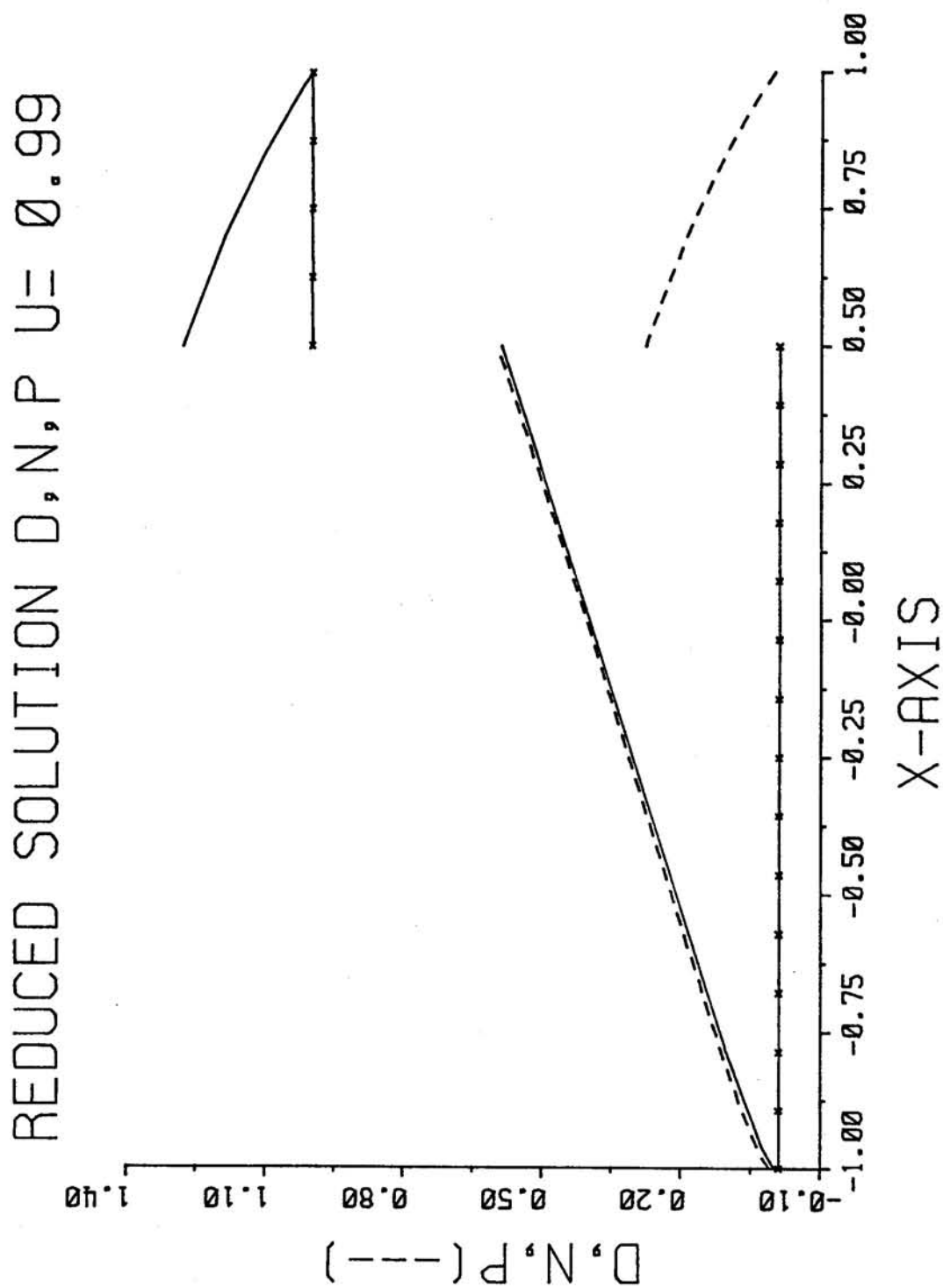


Figure 5

REDUCED SOLUTION PSI     $U = 0.99$

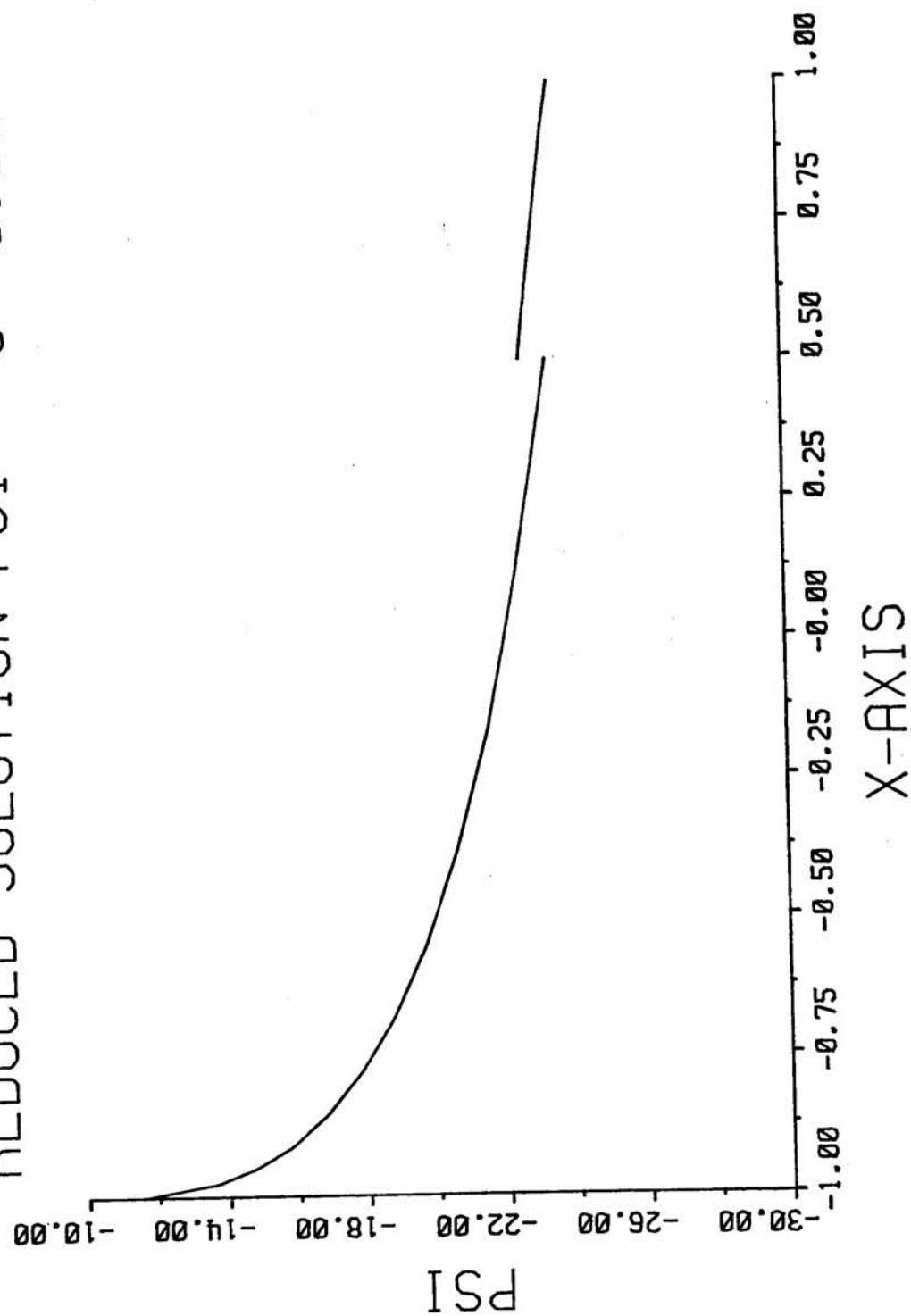


Figure 6

REDUCED SOLUTION JN, JP U= 0.99

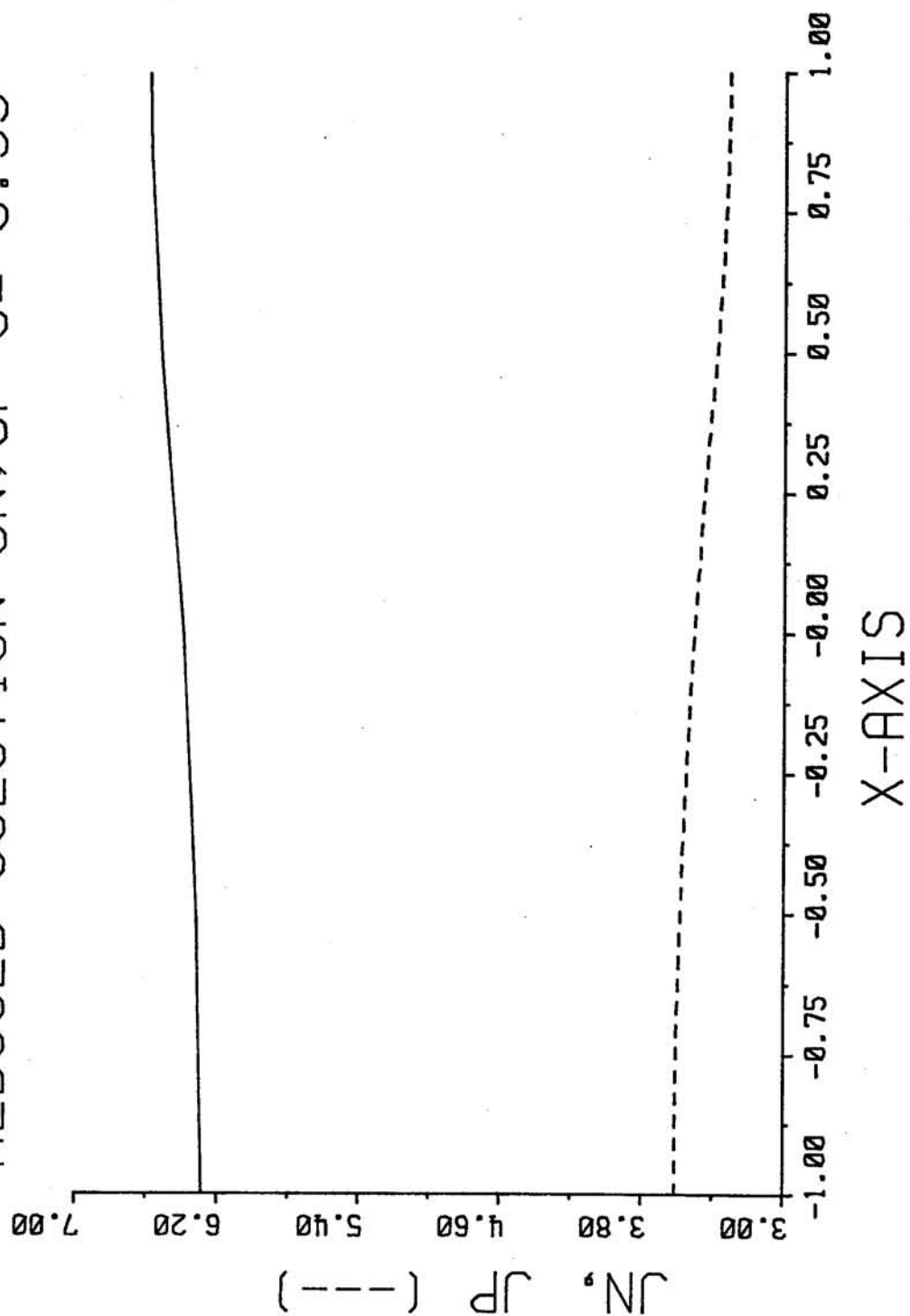


Figure 7  
FULL SOLUTION  $D, N, P$   $U = 0.79$

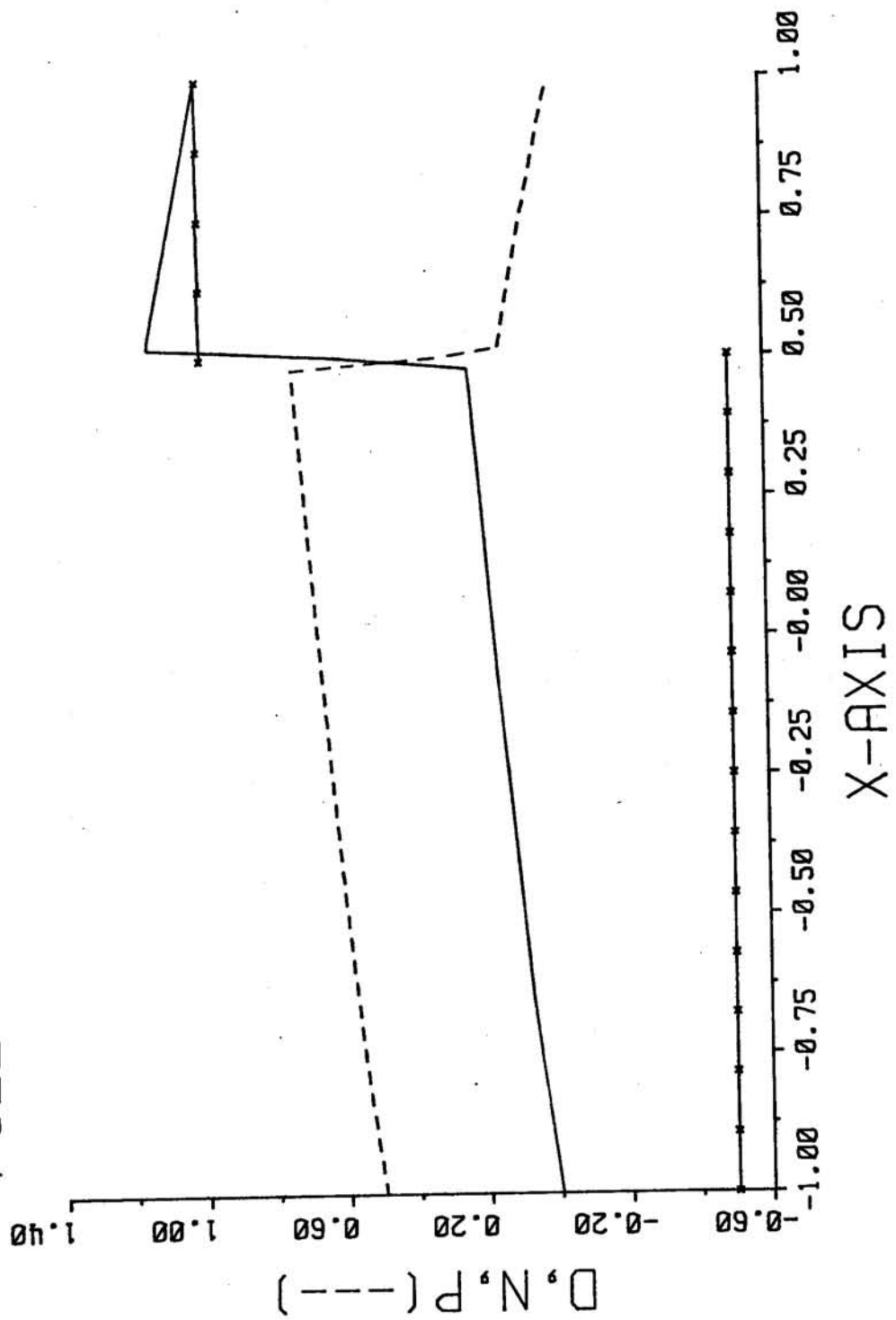




Figure 8

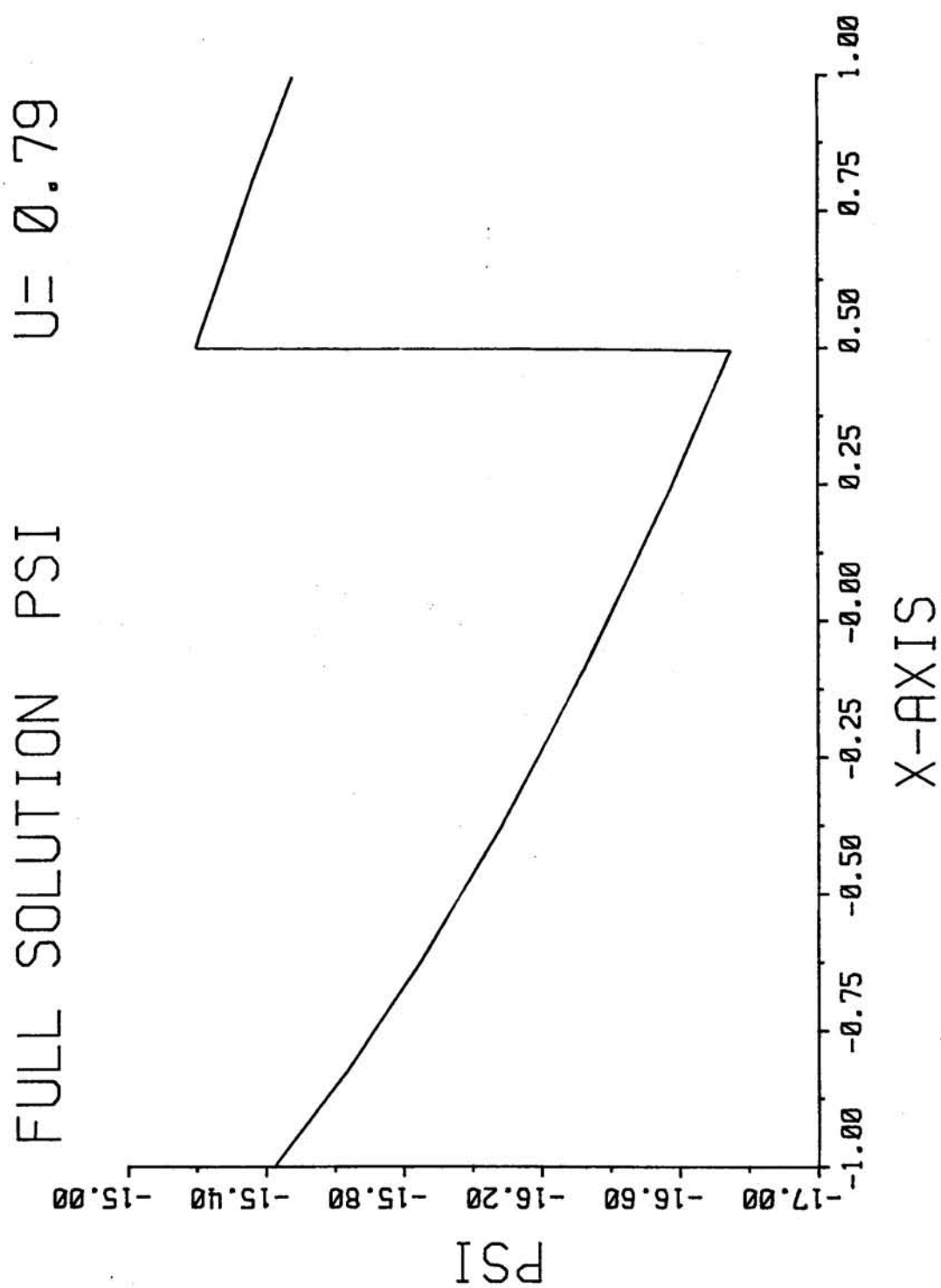


Figure 9

FULL SOLUTION JN, JP U= 0.79

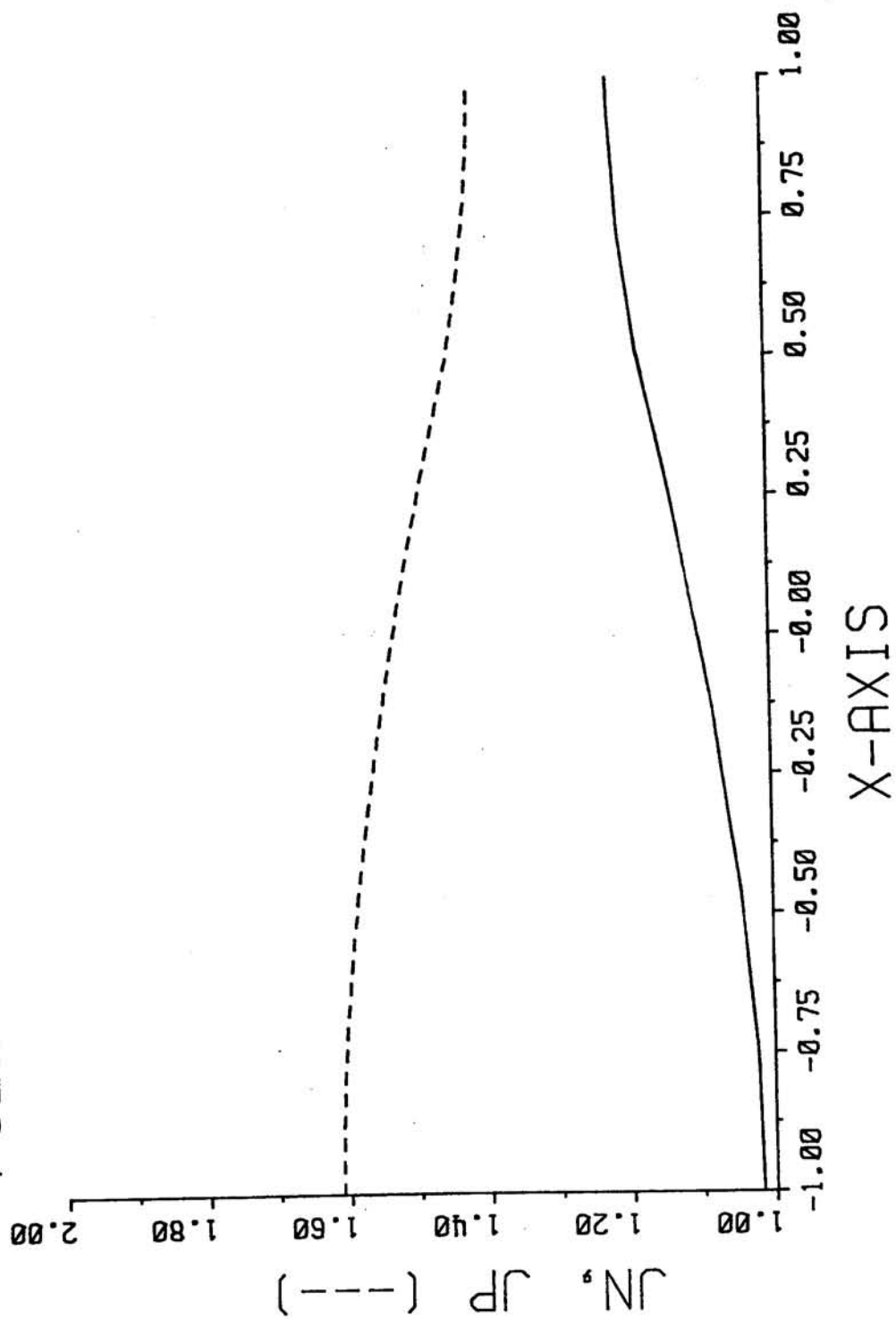


Figure 10

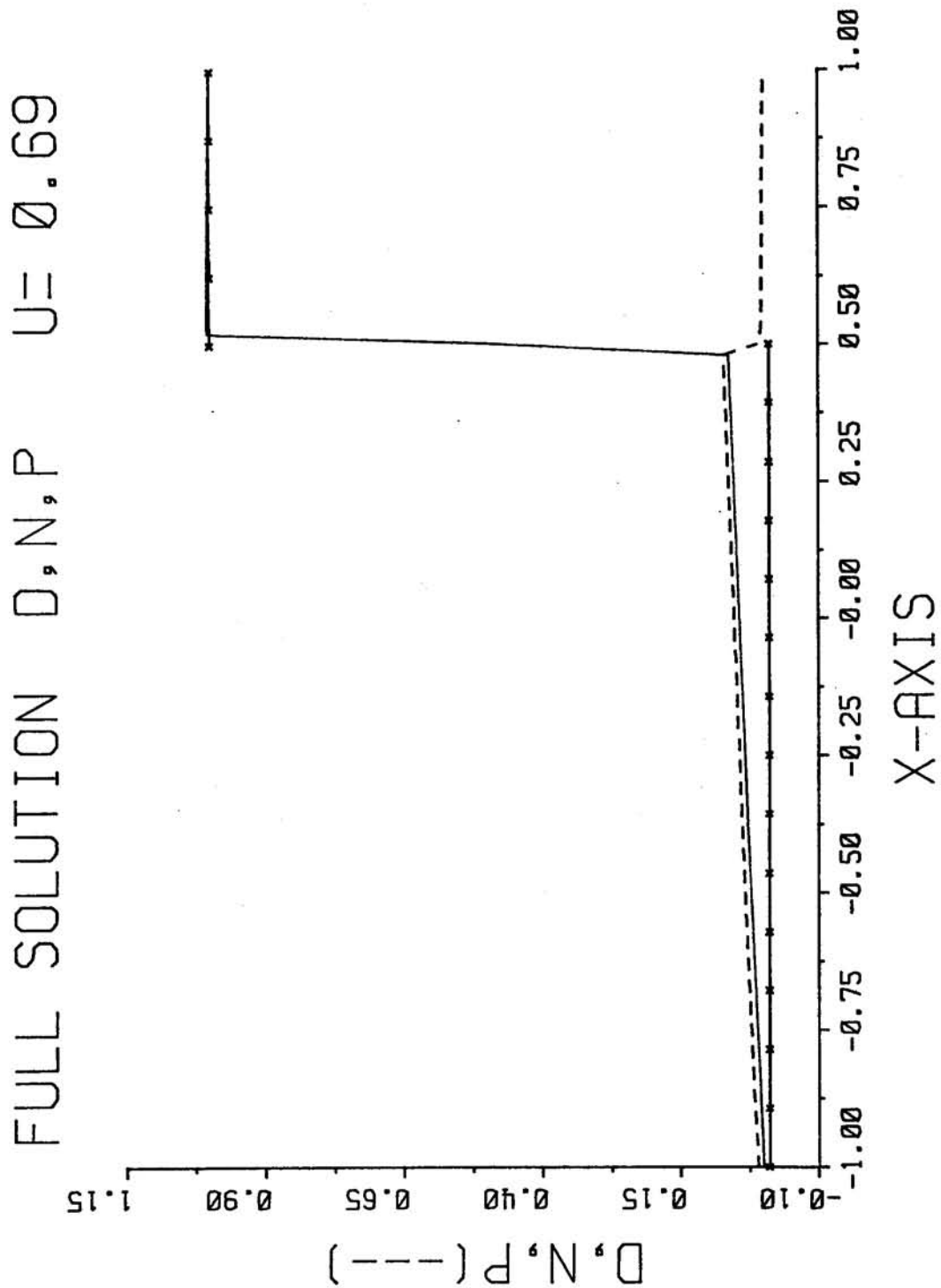


Figure 11

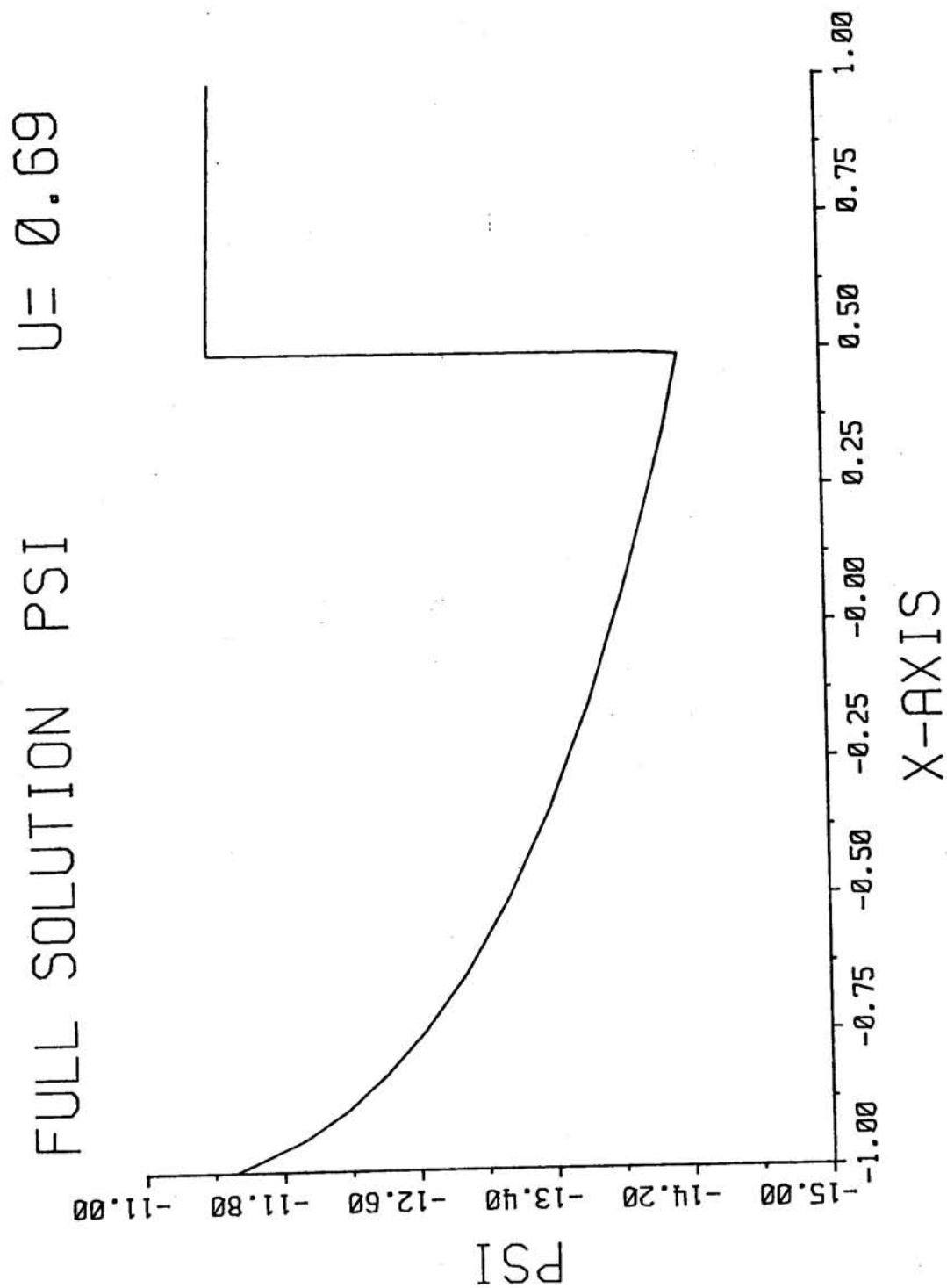


Figure 12

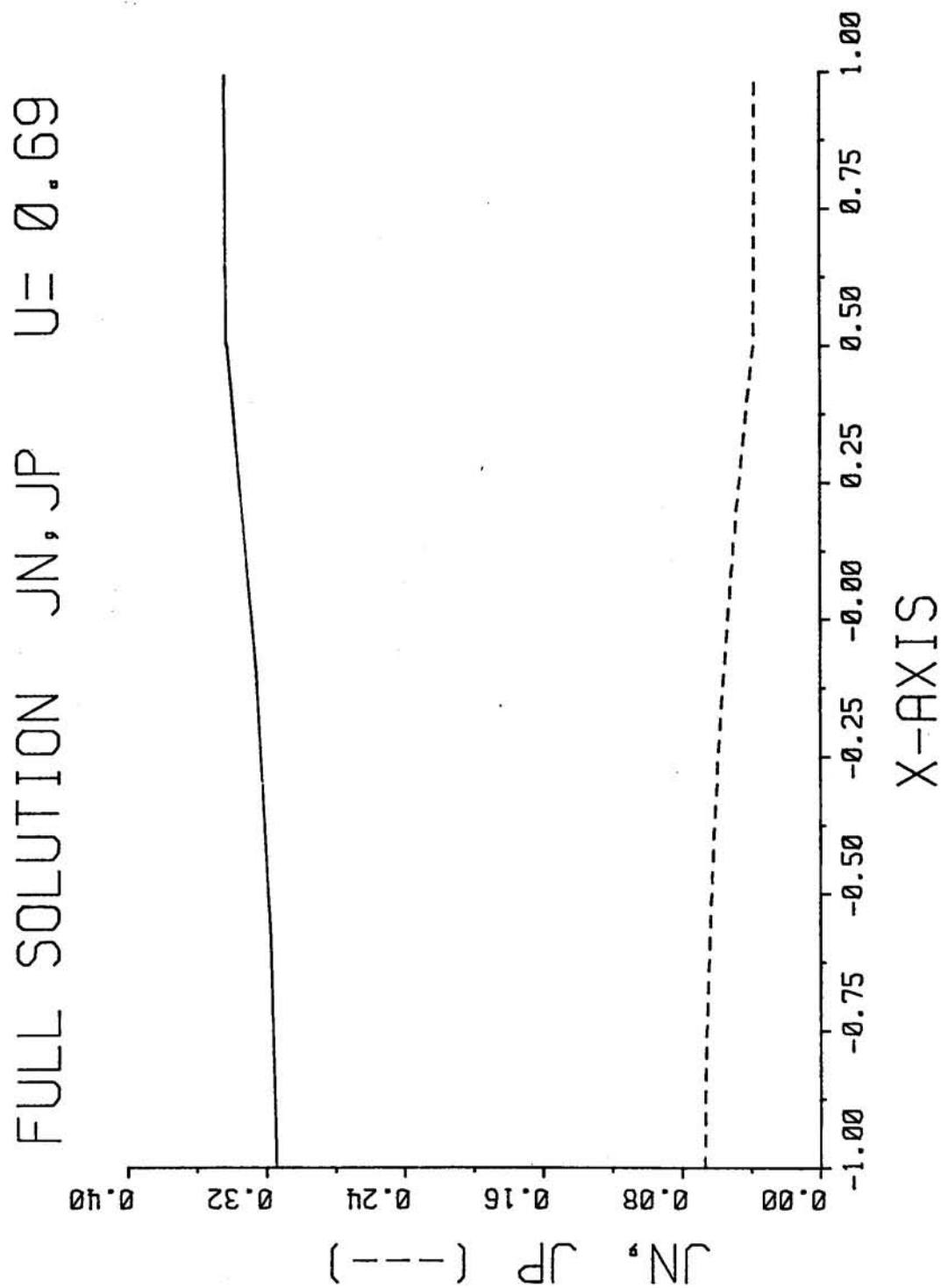
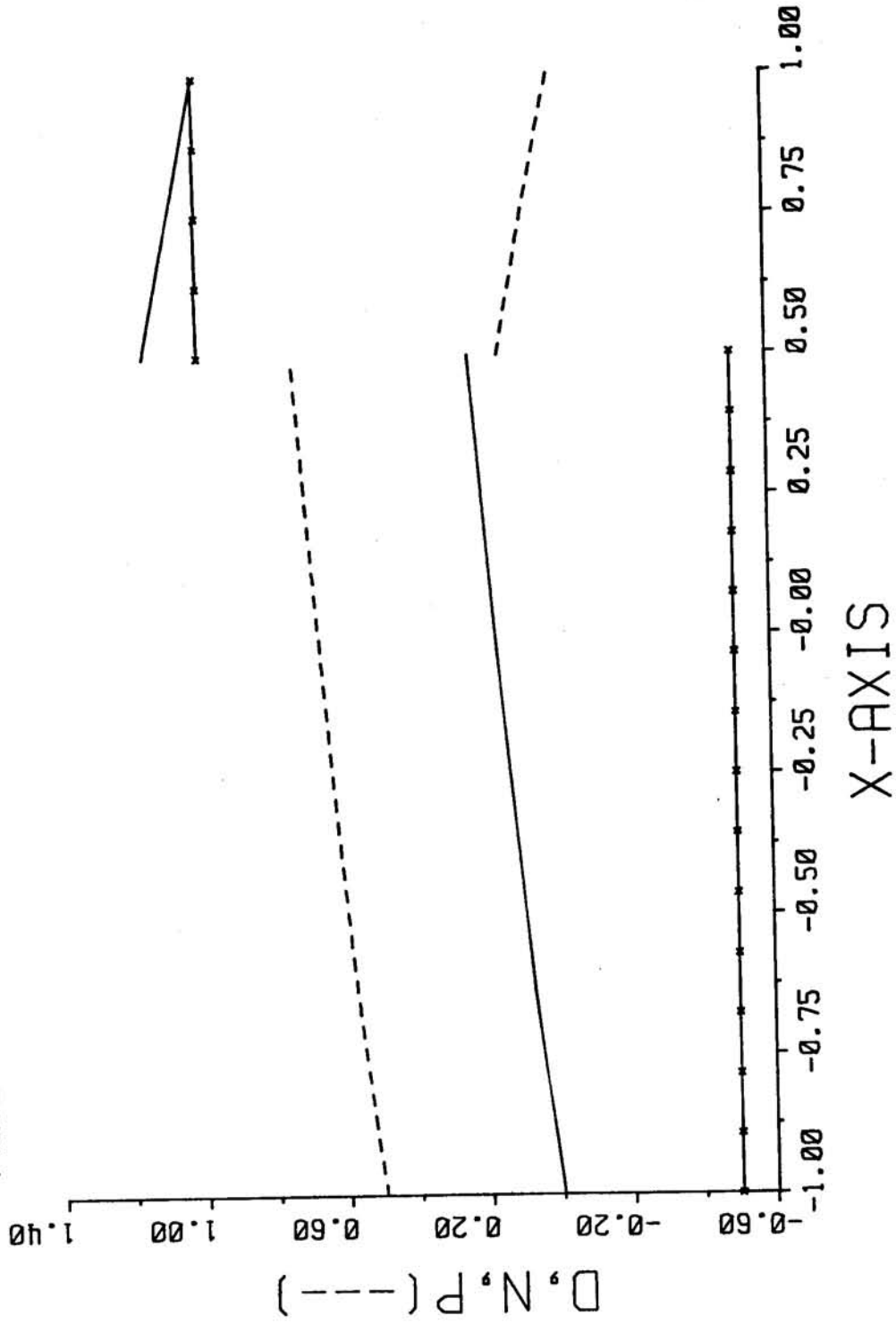


Figure 13

REDUCED SOLUTION  $D, N, P$   $U = 0.79$



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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In this paper we present an analysis of the fundamental one-dimensional semiconductor equations describing potential, carrier, and current density dis- tributions in single-junction semiconductor devices when an external voltage is applied to the contacts. We reformulate the model equations by appropriate scaling as a singularly perturbed two point boundary value problem for a system of nonlinear ordinary differential equations. The right-hand side of the system has a jump discontinuity with respect to the independent variable (space- coordinate) representing the junction between differently doped sides of the		

(continued)



## ABSTRACT (continued)

device. The solution components are assumed to be continuous across this junction.

We give an existence proof for the reduced problem (the singular perturbation parameter is set to zero). The discontinuity of the right-hand side of the system produces a discontinuity in the reduced potential and reduced carrier distributions. This creates an internal layer in the corresponding solution components of the singularly perturbed problem. The current distributions have no internal layer. We also derive the (internal) layer equations and give an existence proof. No boundary layers occur.

We show that formal expansions actually represent (asymptotically) solutions of the singularly perturbed problem if the applied voltage is sufficiently small, and we investigate the dependence of the total current on the applied voltage. Numerical computations are reported.