Two-Dimensional Green's Function of a Semi-Infinite Anisotropic Dielectric in the Wavenumber Domain

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Abstract—**A closed-form expression for the two-dimensional Green's function of a semi-infinite anisotropic dielectric in the wavenumber space is presented, and then the validity and definiteness of the obtained expression for arbitrary values of the wavenumber vector is proven. The derived Green's function is the Fourier transform of the potential response to a point-charge source located on the surface of a constantly stressed semi-infinite anisotropic dielectric. Therefore it is the most significant part in calculating the two-dimensional charge density and field distribution of the surface acoustic wave interdigital transducers in the case of the electrostatic approximation. The integrals associated with the inverse Fourier transform of the derived Green's function are discussed.**

I. Introduction

Using the one-dimensional Green's function in the wavenumber space $\overline{G}(k_x)$, one can show that, under some symmetry conditions [1] or without any kind of restrictions on the geometry and electrical characteristics of the fingers of surface acoustic wave (SAW) interdigital transducers [2], charge density and potential on the surface can be calculated without much effort—including floating fingers if necessary. In the cases of greatest practical interest (aperture-apodized SAW filters, curved-finger SAW filters), the one-dimensional representation and analysis of a SAW filter cannot fulfill the severe conditions of a highly standardized design. In these cases the calculation of two-dimensional charge distribution is indispensible. Following mainly the same formalism as in the one-dimensional case [2], the first problem arising is to find the two-dimensional Green's function in the wavenumber space $\overline{G}(k_x, k_y)$. We derive a closed-form expression for $\overline{G}(k_x, k_y)$, and prove the validity and definiteness of the obtained expression for arbitrary values of the wavenumber vector and discuss the integrals involved in the inverse Fourier transform. From a mathematical point of view, this function is well behaved and, as a consequence of its properties, it is possible to calculate the charge potential interrelation matrix (CPIM) elements [2] in closed form. The calculation of CPIM elements is accompanied by numerous algebraic manipulations, which strongly depend on the form of $\overline{G}(k_x, k_y)$. To emphasize the eminence of the properties of $\overline{G}(k_x, k_y)$ for the evaluation of CPIM elements in closed form, in this paper we restrict ourselves to the derivation and a theoretical discussion of $\overline{G}(k_x, k_y)$.

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II. Theory

We first consider a semi-infinite dielectric with a plane surface and a set of infinitely thin electrodes deposited on it. To facilitate one-dimensional (x) analysis, the fingers may be of infinite length. The electrodes are assumed to have negligible sheet resistivity. Under these conditions the charge density $\sigma(x, \omega)$ and the potential $\varphi(x, \omega)$ at the surface are related by a convolution equation involving the Green's function $G(x)$:

$$\varphi(x, \omega) = \int_{-\infty}^{\infty} G(x - x') \cdot \sigma(x', \omega)\, dx'. \tag{1}$$

Assuming $\varphi(x, \omega)$, $\sigma(x, \omega)$ to vary as $\exp(j\omega t)$ with time t and dropping this factor, we have

$$\varphi(x) = \int_{-\infty}^{\infty} G(x - x') \cdot \sigma(x')\, dx'. \tag{2}$$

The Fourier transform of (2) is [3]

$$\overline{\varphi}(k_x) = \overline{G}(k_x) \cdot \overline{\sigma}(k_x). \tag{3}$$

$G(x)$ is the potential distribution on the surface if there is a line charge source on the surface, neglecting piezoelectricity. $\overline{G}(k_x)$ is the Fourier transform of $G(x)$. Piezoelectricity can be accounted for to some extent using the permitivity constants measured under constant stress ϵ_{ij}^T [4]. In the following the superscript T will be suppressed. The tensor of the permitivity constants (4)

$$(\underline{\epsilon}) = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{pmatrix} \tag{4}$$

is a symmetric positive definite matrix [5]; that is

$$\epsilon_{11} > 0,\ \epsilon_{22} > 0,\ \epsilon_{33} > 0 \tag{5a}$$

$$\epsilon_{11} \cdot \epsilon_{22} - \epsilon_{12}^2 > 0,\ \epsilon_{11} \cdot \epsilon_{33} - \epsilon_{13}^2 > 0,$$

$$\epsilon_{22} \cdot \epsilon_{33} - \epsilon_{23}^2 > 0 \tag{5b}$$

$$\det(\underline{\epsilon}) > 0 \tag{5c}$$

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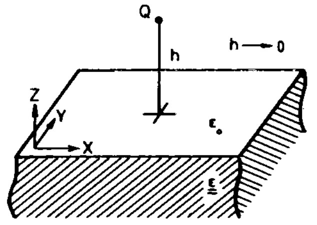

Fig. 1. Point-charge source excitation of a semi-infinite anisotropic dielectric.

The two-dimensional case (3) has the following form:

$$\bar{\varphi}(k_x, k_y) = \bar{G}(k_x, k_y) \cdot \bar{\sigma}(k_x, k_y). \tag{6}$$

$\bar{G}(k_x, k_y)$ is the Fourier transform of the potential distribution on the surface of a constantly stressed semi-infinite anisotropic dielectric, if a point-charge source is located on it (Fig. 1).

By substitution one can show that solutions of the form

$$\varphi(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\varphi}(k_x, k_y) \cdot \exp[jk_x x + jk_y y - j(\epsilon_{13}/\epsilon_{33}k_x + \epsilon_{23}/\epsilon_{33}k_y) \cdot z + [\bar{p}(k_x, k_y)]^{1/2} \cdot z]\, dk_x dk_y \tag{7}$$

with

$$\bar{p}(k_x, k_y) = \begin{pmatrix} k_x \\ k_y \end{pmatrix}^T (\underline{\epsilon}_R) \begin{pmatrix} k_x \\ k_y \end{pmatrix} \tag{8a}$$

$$(\underline{\epsilon}_R) = \begin{pmatrix} \epsilon_{r1}^2 & \epsilon_{r12} \\ \epsilon_{r12} & \epsilon_{r2}^2 \end{pmatrix} \tag{8b}$$

$$\epsilon_{r1}^2 = \epsilon_{11}/\epsilon_{33} - (\epsilon_{13}/\epsilon_{33})^2 \tag{8c}$$

$$\epsilon_{r2}^2 = \epsilon_{22}/\epsilon_{33} - (\epsilon_{23}/\epsilon_{33})^2 \tag{8d}$$

$$\epsilon_{r12} = \epsilon_{12}/\epsilon_{33} - \epsilon_{13}\epsilon_{23}/\epsilon_{33}^2 \tag{8e}$$

and

$$\varphi(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\varphi}(k_x, k_y) \cdot \exp[jk_x x + jk_y y - (k_x^2 + k_y^2)^{1/2} \cdot z]\, dk_x dk_y \tag{9}$$

satisfy the Laplace equation in substrate and vacuum, respectively. The boundary condition at the surface is

$$D_z(x, y, 0^+) - D_z(x, y, 0^-) = \sigma_s(x, y). \tag{10}$$

D_z and σ_s are the z-component of the dielectric displacement $\vec{D}$ and the charge density on the surface, respectively. Using (7), (9), and (10), we obtain

$$\bar{\varphi}(k_x, k_y) = \frac{1}{\epsilon_0 \cdot \sqrt{k_x^2 + k_y^2} + \epsilon_{33} \cdot \sqrt{\bar{p}(k_x, k_y)}} \cdot \bar{\sigma}(k_x, k_y). \tag{11}$$

Equation (11) compared with (6) gives $\bar{G}(k_x, k_y)$. If there is no variation in y-direction ($k_y = 0$), $\bar{G}(k_x, k_y)$ reduces to

$$\bar{G}(k_x) = \frac{1}{(\epsilon_0 + \epsilon_{33} \cdot \epsilon_{r1}) \cdot |k_x|} \tag{12}$$

which is the well-known electrostatic Green's function for a line charge source [6], [7]. An analogous statement is valid for $k_x = 0$. The inversion symmetry of the indices and the variables of (8) should also be noted; that is

$$\bar{G}(1, 2, k_x, k_y) = \bar{G}(2, 1, k_y, k_x). \tag{13}$$

For completeness we show that $\bar{p}(k_x, k_y)$ in (7), (8a), and (11) can not be negative. With $k_x \neq 0$, $k_y \neq 0$ the positive definiteness of $(\underline{\epsilon}_R)$ from (8b) is a sufficient condition for $\bar{p}(k_x, k_y)$ being nonnegative, since $\bar{p}(k_x, k_y)$ is a quadratic-form (8a). Now assume $(\underline{\epsilon}_R)$ is a positive definite matrix, that is

$$\det(\underline{\epsilon}_R) = \epsilon_{r1}^2 \cdot \epsilon_{r2}^2 - \epsilon_{r12}^2 > 0. \tag{14a}$$

The corresponding inequalities for the pair of directions (1, 3) and (2, 3) are

$$\epsilon_{r1}^2 \cdot \epsilon_{r3}^2 - \epsilon_{r13}^2 > 0 \tag{14b}$$

$$\epsilon_{r2}^2 \cdot \epsilon_{r3}^2 - \epsilon_{r23}^2 > 0 \tag{14c}$$

ϵ_{r3}^2, ϵ_{r13}^2, and ϵ_{r23}^2 can be easily constructed from (8c), (8d), and (8e). Adding (14a), (14b), and (14c) after some manipulation gives

$$\epsilon_{11}\epsilon_{22}\epsilon_{33} - \epsilon_{11}\epsilon_{23}^2 - \epsilon_{22}\epsilon_{13}^2 - \epsilon_{33}\epsilon_{12}^2 - 2 \cdot \epsilon_{12}\epsilon_{13}\epsilon_{23} > 0. \tag{15}$$

The left-hand side of the inequality (15) is exactly the determinant of $(\underline{\epsilon})$, which is positive, as stated in (5). The reversed argumentation also is true: (15) splits uniquely into the inequalities (14) because of the inversion symmetry of the indices 1, 2, and 3 in (14) and (15). Therefore (14a) is valid, and it implies the positiveness of $\bar{p}(k_x, k_y)$.

To emphasize more explicitly the eminence of the positiveness of $\bar{p}(k_x, k_y)$, we will briefly discuss the type of the integrals that occur in the inverse Fourier transform from wavenumber space into real space. The latter is a necessary step for solving the electrostatic problems using the elegant formalism of the Green's function in the wavenumber domain. Following mainly the same solution procedure as discussed in [2] we obtain relatively complicated integrals in the (k_x, k_y)-plane. Transformation to a polar coordinate system $(k_x, k_y) \rightarrow (k, \phi)$, considerably simplifies these integrals. The integral in k is of the following form (16):

$$\int_0^\infty \frac{\sin(ka_1)}{ka_1} \cdot \frac{\sin(ka_2)}{ka_2}\, dk \tag{16}$$

with the solution (17), [8]

$$\frac{\pi}{2} \cdot \frac{1}{\max(a_1, a_2)}. \tag{17}$$

a_1 and a_2 are proportional to sin (ϕ) and cos (ϕ), respectively. The integral in ϕ has a more complicated structure. With η = tan (ϕ) and a subsequent effortful algebraic manipulation, one can show that this integral reduces to a sum of either some integrals that can be directly evaluated or integrals which can be reduced to the following type (18):

$$\int_{\eta_1}^{\eta_2} \frac{m \cdot \eta + n}{p_{2,1}(\eta) \cdot \sqrt{p_{2,2}(\eta)}}\, d\eta \tag{18}$$

depending on the sign of

$$\Delta = b_2^2 - (b_1 - 1) \cdot (b_3 - 1)$$

where

$$b_1 = \epsilon_{r1}^2, \quad b_2 = \epsilon_{r12}, \quad b_3 = \epsilon_{r2}^2.$$

ϵ_{r1}^2, ϵ_{r12}, and ϵ_{r2}^2 are the elements of ($\underline{\epsilon}_R$) in (8b). Integrals of the type (18) can be solved in closed form [8]. Then $p_{2,1}(\eta)$ and $p_{2,2}(\eta)$ in (18) are

$$p_{2,1} = (b_1 - 1) \cdot \eta^2 + 2b_2 \cdot \eta + (b_3 - 1)$$

$$p_{2,2} = b_1 \cdot \eta^2 + 2b_2 \cdot \eta + b_3.$$

$p_{2,2}(\eta)$ directly reflects the form of $\overline{p}(k_x, k_y)$. In fact $p_{2,2}(\eta)$ can be obtained formally from $\overline{p}(k_x, k_y)$ by the substitution $(k_x, k_y) \rightarrow (\eta, 1)$. Therefore $p_{2,2}(\eta)$ is a quadratic-form. That means $p_{2,2}(\eta) > 0$ for arbitrary values of η. The solution of (18) generally contains radicals of positive second-order polynomials, arsh ($f(\eta)$) and log ($g(\eta)$) with well-behaved functions $f(\eta)$ and $g(\eta)$. Note that an arbitrary change of the variable in (18), that is $\eta = h(\overline{\eta})$, does not injure the positiveness property of $p_{2,2}(\eta)$. Exactly this attribute of $p_{2,2}(\eta)$ makes it possible to transform the integrals to types which are easier to handle. A linear composition of the results of the above mentioned integrals determines in a more or less trivial but effortful manner the charge potential-interrelation matrix (CPIM) elements [2].

III. Conclusion

We have derived a closed-form expression for the two-dimensional Green's function of a semi-infinite anisotropic dielectric in the wavenumber space and proved the validity and definiteness of the derived expression for arbitrary values of the wavenumber vector. It is shown that this function is well behaved, with the pleasant consequence that the integrals occurring in the inverse Fourier transform can be expressed in closed form. The integrals associated with the inverse Fourier transform of the derived Green's function are discussed.

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References

[1] V. M. Ristic and A. Hussein, "Surface charge and field distribution in a finite SAW transducer," *IEEE Trans. Microwave Theory Tech.*, 1979, vol. MTT-27, pp. 897-901.

[2] A. R. Baghai-Wadji, S. Selberherr, and F. Seifert, "On the calculation of charge, electrostatic potential and capacitance in generalized finite SAW structure," in *Proc. IEEE Ultrason. Symp.*, 1984, pp. 44-48.

[3] D. C. Champeney, *Fourier Transform and Their Physical Application.* New Jersey: Academic, 1973, pp. 15-20.

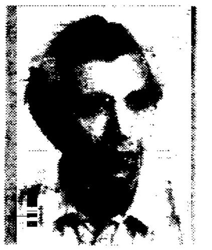
[4] B. A. Auld, *Acoustic Fields and Waves in Solids.* New York: Wiley-Interscience, 1973, vol. 1, pp. 380-381.

[5] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations.* New Jersey: Prentice-Hall, 1967, pp. 56-61.

[6] R. F. Milsom, N. H. C. Reilly and M. Redwood, "Analysis of generation and detection of surface and bulk acoustic waves by interdigital transducers," *IEEE Trans. Sonics Ultrason.*, vol. SU-24, pp. 147-166, 1977.

[7] D. P. Morgan, "Quasi-static analysis of generalized SAW transducers using the Green's function method," *IEEE Trans. Sonics Ultrason.*, 1980, vol. SU-27, pp. 111-123.

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products.* New York: Academic, 1980, pp. 80-81, 414.



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