

APPLIED PHYSICS REVIEWS—FOCUSED REVIEW

Recent advances in Wigner function approaches

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The Wigner function was formulated in 1932 by Eugene Paul Wigner, at a time when quantum mechanics was in its infancy. In doing so, he brought phase space representations into quantum mechanics. However, its unique nature also made it very interesting for classical approaches and for identifying the deviations from classical behavior and the entanglement that can occur in quantum systems. What stands out, though, is the feature to experimentally reconstruct the Wigner function, which provides far more information on the system than can be obtained by any other quantum approach. This feature is particularly important for the field of quantum information processing and quantum physics. However, the Wigner function finds wide-ranging use cases in other dominant and highly active fields as well, such as in quantum electronics—to model the electron transport, in quantum chemistry—to calculate the static and dynamical properties of many-body quantum systems, and in signal processing—to investigate waves passing through certain media. What is peculiar in recent years is a strong increase in applying it: Although originally formulated 86 years ago, only today the full potential of the Wigner function—both in ability and diversity—begins to surface. This review, as well as a growing, dedicated Wigner community, is a testament to this development and gives a broad and concise overview of recent advancements in different fields. © 2018 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>). <https://doi.org/10.1063/1.5046663>

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I. INTRODUCTION

The years after the arrival of quantum mechanics around 1900 challenged the existing understanding of physics in many aspects. The established wave picture of light was put into question by Planck¹ and Einstein,² who demonstrated the particle nature of light. This was followed by Thomson³ and Taylor⁴ showing that indeed light behaves as particles and waves. This duality has then been also established for particles by Bohr⁵ and de Broglie.⁶ Heisenberg⁷ introduced a new view of quantum mechanics, which together with Born and Jordan yielded a new mathematical formulation.^{8,9} Based on de Broglie's work, Schrödinger developed his wave mechanics around his famous equation,^{10–14} which was soon extended by Madelung¹⁵ and Kennard¹⁶ as well as much later by Bohm.^{17,18}

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Eugene Paul (*Hungarian* Jenô Pál) Wigner, concerned with the problem of a phase space representation, found a way to transform the wave function to introduce the missing momentum variables and published the result in 1932.¹⁹ Wigner suggested that one can consider the following expression for the probability in phase space:

$$P(x, p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} dy \psi^*(x+y) \psi(x-y) e^{\frac{2iy p}{\hbar}}, \quad (1)$$

which today is termed the Wigner function. Here, we have recovered a phase space distribution function which is based upon the quantum wave function. However, as Wigner noted, this function is real but is not always positive definite. Wigner was primarily interested in the corrections to the description of thermodynamic equilibrium that would arise from the presence of quantum mechanics. To this end, he also found a correction to the energy, specifically to the potential energy. This, ultimately, means that the Wigner approach also leads to a quantum correction to the potential that has similar form to that found by Madelung¹⁵ and Bohm.^{17,18} The difference is in the magnitude of this correction, but the form appears to be very similar. Important to the success of the Wigner function, over a decade later, the achievements of Groenewold in 1946^{20,21} and Moyal in 1949^{21,22} independently lead to a complete formalism where the time evolution of a quantum system itself was defined in phase space alone (rather than in Hilbert space).^{21,23–26}

The important result of Wigner's introduction of the phase space distribution is that it is now possible to treat classical mechanics and thermodynamics on an equal footing with quantum mechanics and quantum thermodynamics. This introduces an additional window into the quantum world and makes it easier to discover the novel new effects of quantum mechanics. Such a phase space approach has become far more useful in recent years where one wants to see the quantum corrections and to make them visible in the analysis of experiments.

A. Extending the Wigner function and its applications

The Wigner distribution function is not the only quantum phase space distribution.^{27–31} In 1940, Husimi published another famous distribution.³² The Husimi distribution function is basically a smoothed version of the Wigner function, as it is a positive semi-definite and semi-classical distribution. The Husimi distribution thus is a coarse-graining of the Wigner function with a Gaussian smoothing.³³ As a result, the Husimi function does not produce the correct probability functions.³⁴ Regardless, the use of the Husimi function proved very beneficial to applications in quantum optics and quantum chaos.³⁵ Although the Husimi distribution function is perhaps the best known distribution aside from the Wigner function itself, other distribution functions exist as well.^{31,36}

An alternative view of quantum mechanics has been established by Bohm in 1952, which specifically incorporates particle trajectories.^{17,18} Here, the Bohm potential provides an addition to the total energy and provides a non-classical force which guides the wave functions in a

self-consistent manner. This can provide a basis to use a particle representation, where the particles move through the presence of both the classical and the quantum forces. The approach was initially based on a *hidden variable* theory, which was later adapted to the *ontological theory* by Bohm and Hiley.³⁷ Based on Bohmian mechanics, a phase-space picture of quantum mechanics with well-defined position and momentum becomes a *natural* construction and offers an interesting alternative to the Wigner picture.^{28,38}

In the decades to follow Groenewold and Moyal, the development of the Wigner function and its use continued (for overviews see, for instance, Refs. 21, 26, and 39–47). It must be stressed that in the second half of the 20th century (in particular, towards the end), this phase space distribution found its use in a broad range of applied areas in science and engineering. Here, a short, whirlwind overview of the many developments which started in this period is given, laying the groundwork for discussing most recent advances in Secs. II–VII.

Among the first adopters of the Wigner function was optics: In 1965, the quantum phase space distribution was used to describe the coherence of optical fields and to describe the polarization of these fields.⁴⁸ In 1977, the semi-classical limit was explored.⁴⁹ In the early 1980s, the Wigner function was utilized to investigate quantum effects in electron transport.^{50,51} In 1984, a particularly important review—with Wigner as co-author—on the fundamental properties of the Wigner distribution function and on other distribution functions was published.³¹ In this period, the Wigner function has also been investigated for use in elementary pattern recognition.⁵² Then, in the mid-late 1980s, the use of the Wigner function to describe transport in resonant tunneling devices began to be explored,^{53–55} and this approach for device transport grew over the subsequent decades.^{56,57} In the 1990s, again with Wigner as a co-author, the wave-packet spread, coherent-state representation, and squeezed states of light began to be investigated for quantum optics.⁵⁸ Also, the Wigner function started to gain traction in the field of chemistry, where it was used to study the dynamics of tunneling, resonance, and dissipation in the transport across/through a barrier⁵⁹ as well as (later in 2003) decoherence.⁶⁰ In this period, its use appeared for wave propagation through media⁶¹ and for wavelets.⁶² Furthermore, the Wigner function was used to investigate different theories of quantum dissipation⁶³ and the mathematical basis of the time-independent Wigner functions was explored.⁶⁴ In 2001, the Wigner function was also applied to economics.⁶⁵ Then, it became clear at the turn of the century that the Wigner function was perfect for illustrating the quantum property of entanglement, which became exceedingly important in quantum information theory.^{66–81} In conclusion, by the end of the 20th century, the peculiar properties of the Wigner function found a plethora of applications in a diverse set of research areas, providing a fertile ground for follow-up research in the 21st century.

B. What this review covers

In Secs. I and I A, we described the arrival of the Wigner function and its development throughout the 20th

century. The built-up momentum did not stop there: Since the 2000s the Wigner function found increased use in many application areas. In some fields, it is the main method of choice, whereas in others it continues to play a side role. Regardless, this increased interest beyond a single field gave rise to the Wigner Initiative [<http://www.iue.tuwien.ac.at/wigner-wiki/>] and with it, additional efforts to introduce cohesion into the growing, diverse Wigner community: A special issue in the Journal of Computational Electronics⁸² was published and the International Wigner Workshop (IW²) series was established. The latter was begun in 2015 [<https://www.asu.edu/aie/ISANN15/WignerWorkshop.htm>], with the second installment in 2017 [<http://www.iue.tuwien.ac.at/iw22017/>], and the third one planned for 2019 [<http://www.iue.tuwien.ac.at/iwcn2019/workshops/iw22019/>]. This review is testament to this development and summarizes the versatility and far-reaching applicability as well as the high pace of research in many different fields which depend on the Wigner function.

Section II will set the stage by discussing the general properties of the Wigner function, highlighting its use for depicting entanglement—a key property. Beginning with Sec. III, recent advances (with the vast majority not older than 2015) in using the Wigner function in many different application fields are discussed.

In quantum information processing (cf. Sec. III), the Wigner function finds multiple uses, ranging from reconstruction to enable the detection of squeezed states and highlighting entanglement to using the negativity for identifying non-classicality (or *quantumness*). Those fundamental properties are, in particular, important for optical qubits and thus for quantum information.

In quantum physics (cf. Sec. IV), the Wigner function is regularly employed to investigate the transition from a quantum to a classical regime (i.e., decoherence processes) and again to highlight entanglement. In particular noteworthy is the application to superconductivity, i.e., Josephson junctions and qubits.

In quantum electronics (cf. Sec. V), the Wigner function is used to model quantum electron transport, for which extensive efforts were put into tackling the numerical and computational challenges as well as into integrating scattering processes and electromagnetic fields into the modeling approaches. Nowadays, realistic devices like the resonant tunneling diode are (still) being investigated as are transport-confined devices and exploratory new devices and systems, also within the area of entangletronics.

In quantum chemistry (cf. Sec. VI), the Wigner function is primarily used to investigate the spectra of molecules and atoms, tunneling effects, and certain quantum correlations. Nevertheless, the typical models used are found in many quantum physics applications.

In signal processing (cf. Sec. VII), the Wigner function is a key tool to investigate wave propagations through certain media, giving insight into the nature of the materials the waves pass through. Also, they are used, in practice, to describe the properties of the signals themselves. In particular, classical wavelets have a strong connection to Wigner functions.

All in all, what will be seen from the extensive literature review is that the areas with the most focused Wigner activities are in quantum information and devices, such as optical qubits (cf. Sec. III) and superconductor qubits (cf. Sec. IV). If there is one take-away message for the reader, it is that Wigner functions have become very important because, if for no other reason, they can be reconstructed from experimental measurements in a manner that illustrates the clear presence of entanglement in the quantum system. This seems to be unique in quantum transport.

II. WIGNER FUNCTION AND ENTANGLEMENT

As previously indicated in (1), the Wigner function has a convenient phase space formulation. Wigner gave us his formulation already in 1932 and demonstrated its usefulness when spanning the transition from the classical to the quantum world. The Wigner function approach is felt to offer a number of advantages for use in modeling the behavior of physical processes in many fields of science. First, it is a phase-space formulation, similar to classical formulations based upon the Boltzmann equation. By this, we mean that the Wigner function involves both real space and momentum space variables, distinctly different from the Schrödinger equation. In this regard, modern approaches with the Wigner function provide a distinct formulation that is recognized as equivalent to (but is a different alternative to) normal operator-based quantum mechanics. Because of the phase-space nature of the distribution, it is now conceptually possible to identify where quantum corrections enter a problem by highlighting the differences from the classical version, an approach that has been used to provide an effective quantum potential that can be used as a correction term in classical simulations.

In the following, we will provide a concise derivation of the Wigner equation (the Wigner equation is often referred to as the Wigner transport equation). We start with the density matrix—providing a convenient description of mixed states—which can be derived from the Schrödinger picture

$$\rho(\mathbf{r}, \mathbf{r}', t) = \sum_{n,m} c_{nm} \phi_m^*(\mathbf{r}, t) \phi_n(\mathbf{r}', t) \equiv \psi^*(\mathbf{r}, t) \psi(\mathbf{r}', t), \quad (2)$$

where c_{nm} is the statistical weight and ϕ_n and ϕ_m^* are basis functions. The center-of-mass coordinates, the average (\mathbf{r}) and difference (\mathbf{s}) coordinates, are introduced as

$$\mathbf{x} = \frac{\mathbf{r} + \mathbf{r}'}{2}, \quad \mathbf{s} = \mathbf{r} - \mathbf{r}'; \quad \mathbf{r} = \mathbf{x} + \frac{\mathbf{s}}{2}, \quad \mathbf{r}' = \mathbf{x} - \frac{\mathbf{s}}{2}, \quad (3)$$

so that

$$\rho(\mathbf{r}, \mathbf{r}', t) = \rho\left(\mathbf{x} + \frac{\mathbf{s}}{2}, \mathbf{x} - \frac{\mathbf{s}}{2}, t\right). \quad (4)$$

Using the above center-of-mass coordinates, the phase-space Wigner function can be represented as the Fourier transform of the density matrix on the difference coordinate \mathbf{s}

$$f_W(\mathbf{x}, \mathbf{p}, t) = \frac{1}{h^3} \int d\mathbf{s} \rho\left(\mathbf{x} + \frac{\mathbf{s}}{2}, \mathbf{x} - \frac{\mathbf{s}}{2}, t\right) e^{-i\mathbf{p} \cdot \frac{\mathbf{s}}{\hbar}}, \quad (5)$$

where the density matrix ρ is expressed in terms of the wave functions.^{19,39} This transform, although introduced by Wigner, is often called the Weyl transform.^{19,83,84} Incorporating the center-of-mass coordinate transformations (3) into the Liouville-von Neumann equation yields a new equation of motion for the density matrix

$$i\hbar \frac{\partial \rho}{\partial t} = \left[-\frac{\hbar^2}{m^*} \frac{\partial^2}{\partial \mathbf{x} \partial \mathbf{s}} + V\left(\mathbf{x} + \frac{\mathbf{s}}{2}\right) - V\left(\mathbf{x} - \frac{\mathbf{s}}{2}\right) \right] \rho(\mathbf{x}, \mathbf{s}, t). \quad (6)$$

When the Weyl transform is applied to this equation, one gets an equation of motion for the Wigner function as

$$\frac{\partial f_W}{\partial t} + \frac{\mathbf{p}}{m^*} \cdot \frac{\partial f_W}{\partial \mathbf{x}} - \frac{1}{i\hbar} \left[V\left(\mathbf{x} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}\right) - V\left(\mathbf{x} - \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}\right) \right] f_W = 0. \quad (7)$$

In the absence of dissipative processes, this can be rewritten in a more usable form (henceforth referred to as the Wigner equation)

$$\frac{\partial f_W}{\partial t} + \frac{\mathbf{p}}{m^*} \cdot \frac{\partial f_W}{\partial \mathbf{x}} - \frac{1}{\hbar^3} \int d^3 p' W(\mathbf{x}, \mathbf{p}') f_W(\mathbf{x}, \mathbf{p} + \mathbf{p}') = 0, \quad (8)$$

where

$$W(\mathbf{x}, \mathbf{p}') = \int d^3 s \sin\left(\frac{\mathbf{p}' \cdot \mathbf{s}}{\hbar}\right) \left[V\left(\mathbf{x} + \frac{\mathbf{s}}{2}\right) - V\left(\mathbf{x} - \frac{\mathbf{s}}{2}\right) \right] \quad (9)$$

is the Wigner potential. In the latter term, we can clearly see the nonlocal behavior of the potential that is unique to the system. If the potential is of quadratic, or lower order behavior, then the results are those found with the Boltzmann equation. That is, a simple quadratic potential yields only the first derivative of the potential in (9). This is the electric field, which is the principle driving force for transport, even in the classical case. On the other hand, if the potential is sharp, such as a step in potential, then its effect is felt to all orders of derivatives and can lead to difficulties in numerical simulations.

The problem with even the equation of motion (7) is that one must know the value of the Wigner function, or the wave function, at $t = 0$. Hence, one must also solve the adjoint equation for the initial condition.⁴³ The importance of the initial conditions is especially critical for what is known as the bound state problem. For example, if the confining potential in (9) is just a quadratic potential, it represents the simple harmonic oscillator, and the resulting force from (9) is purely classical. The quantum bound states of the potential will not result from Eq. (7). The initial states from the adjoint equation are required to provide the quantization and, therefore, these bound states.⁴³

The nonlocal behavior of the potential is reflected in just how the wave function at one point senses the behavior of the wave function at a distant point. In other words, there is a

correlation in the wave function between two points if these two points are widely separated. It turns out that this effect can be significant in the absence of decoherence processes, and the correlation length can be quite large.

An important consideration is that the Wigner function satisfies the normalizations that are required on either real space wave functions or momentum space wave functions.³⁹ That is, we obtain the probability of finding the *particle* at any position in space by integrating the Wigner function over the momentum, which gives the magnitude squared of the wave function itself. When integrating the Wigner function over all positions, we find the normalization in the momentum space. This is an important result, as not all suggested phase space formulations of quantum transport necessarily preserve this normalization. This has an important generalization. If we have a function $F(x, p)$, which is either a function of the position operator alone or of the momentum operator alone, or of any *additive* combination of these two operators, the expectation value is completely analogous to the equivalent classical expression for such an average.

Another important feature is that the Wigner function does not spread with time for a fixed momentum, even though the underlying wave function does so.⁴³ That is, in normal quantum mechanics, a Gaussian wave packet in position alone that moves with such a velocity spreads out in a diffusive manner, as the Schrödinger equation is technically a diffusion equation. In some sense, the spread in velocities leads to an increasing uncertainty for the normal Gaussian packet. This does not seem to be the case for the Wigner function, however. Moreover, the corresponding uncertainties satisfy $\Delta x \Delta p_x \geq \hbar/2$ in phase space. Thus, the Wigner function for a single free particle is already different from the classical delta function as a result of the quantum uncertainty relation, and the *size* of the particle is correspondingly much larger due to this uncertainty. That being said, because the uncertainty relation means that we really cannot have a single fixed momentum, or position, the spread in momentum will lead to a spreading of the Wigner function. The sharper the spread in momentum, the broader the spread in position will be, as these are coupled. While we will not give the formal derivation of the equation of motion, it has been derived by many different approaches, and the reader can find one such approach from Ref. 39.

One of the interesting aspects of the Wigner function is the ability to transfer many of the results of classical transport theory into quantum approaches merely by replacing the Boltzmann distribution with the Wigner function. However, there is a caveat that must be understood in this regard. And, this is the fact that the Wigner function may not be a positive-definite function. That is, there are cases in which the Wigner function possesses negative values, and these values can extend over a phase space region of the extent of Planck's constant. If one smooths over a region of this size, the negative excursions will go away. Clearly, these negative excursions represent the appearance of uncertainty in the quantum realm but also have other important properties which will be highlighted in the subsequent application sections. Indeed, this negativity is important in many fields for

signifying the presence of non-Gaussian properties and especially for the existence of entanglement.

A. Entanglement

Entanglement surfaced in 1935 with the famous *thought experiment* of the Einstein-Podolsky-Rosen (EPR) paradox,⁸⁵ although it was Schrödinger who coined the term *entanglement*.⁸⁶ In essence, if two separate quantum bodies interact, they form a single quantum system, in which each body leaves traces of itself on the other body—they become *entangled*. Once separated, the two bodies still remain a single system. The maximal knowledge of this entire single entity is likely more than the available knowledge of the original two bodies. This *memory of the interaction*—the entanglement—persists regardless of the distance between the bodies until some form of *decoherence* happens, which destroys the *memory* and thus breaks the entanglement of the two bodies.

To illustrate entanglement, let us consider a wave function consisting of two Gaussian wave packets which have interacted and thus formed a composite single wave function^{27,39}

$$\psi_T(x, t) = \frac{1}{\sqrt{2}} [\psi_1(x - x_0, k_0) + \psi_2(x + x_0, -k_0)]. \quad (10)$$

One portion of the wave function is centered at x_0 and moves in the positive x direction, and a second portion is centered at $-x_0$ and moves in the negative x direction. When the Wigner function from such a composite wave function is generated, an extra term arises³⁴

$$f_W(x, p, t) = \frac{1}{4} [f_W^+(x, p, t) + f_W^-(x, p, t) + 2e^{-\frac{x^2}{2\sigma^2} - 2\sigma^2 p^2} \cos(2k_0 x)]. \quad (11)$$

The last term in the brackets is a new term that represents the memory interference that arises from the entanglement between the two wave packets.⁸⁷ This term is centered around $x = p = 0$ and oscillates rapidly along the x axis. In Fig. 1, we plot the Wigner function of (11) at a time for which the two original packets have moved away from $x = 0$. The central entanglement term is still seen to be present.

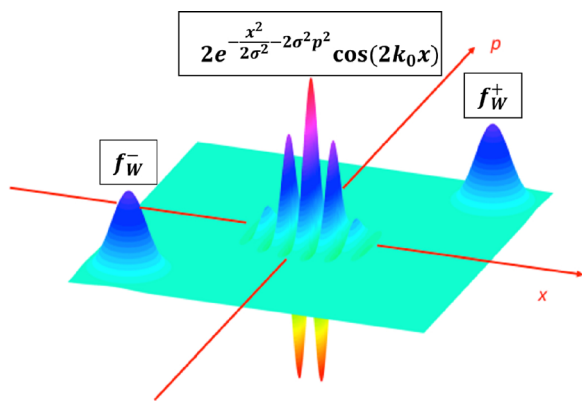


FIG. 1. Two Gaussian wave packets are shown, under the assumption that they form a single wave function. Depicted is the resulting Wigner function, with the entanglement exhibited around $x = p = 0$.

While this is a somewhat trivial example, it is precisely the form that is usually constructed as a *Schrödinger Cat state*, with which one is particularly interested to study the entanglement.^{67,88} In this view, the two Gaussians are the *live* and *dead* Cat states, and the third term is the *alive or dead* entanglement. We will meet these states in later sections.

Naturally, the question of measuring entanglement arises, which, however, is strongly related to how entanglement is defined. Arisen from the EPR paradox and coined by Schrödinger, Bohm described entanglement based on splitting a diatomic molecule, which is split apart by a method that does not affect the spin of each of the two atoms.^{17,18,89}

In recent years, Bohm's approach established itself as the *de facto* wave function used for the EPR paradox. However, there are some problems. For instance, the positions and the momenta of the two atoms must be considered as *hidden* variables, as no experiment on the system can yield any information about the probability distribution of these quantities. This gives rise to an important advantage of the Wigner function: It provides a phase space representation of position *and* momentum and has been shown to describe the existence of the entanglement at the same time. This aspect is particularly relevant to the field of quantum information processing, which lead to a plethora of various measures and use cases of entanglement.⁶⁶⁻⁸¹

III. QUANTUM INFORMATION PROCESSING

Quantum information processing is an umbrella term and, in general, deals with adding quantum effects to the field of information processing. This has many sub-branches, among those are quantum optics, quantum computing, quantum entanglement, quantum teleportation, and quantum metrology. Here, quantum optics stands out as it was one of the early adopters of the Wigner function.⁹⁰⁻⁹² However, Wigner functions also found wide spread use in classical optics over the last decades:⁹³⁻⁹⁷ More recently, they were used for designing new optical achromatic lenses with sub-wavelength focusing⁹⁸ and for optical nanoantenna arrays.⁹⁹

In this section, we are providing an overview of the many uses cases of the Wigner function in quantum information processing. We categorized the contributions based on their individual underlying and predominant physics, revealing that the same concepts find their applications in different fields.

A. The Jaynes-Cummings model

The coupling between a two-level atom and a resonant (quantized) optical cavity is described by a relatively old model, termed as the Jaynes-Cummings model.¹⁰⁰ The model describes the interaction of the two-level atom as it interacts with a quantized mode of an optical cavity and is of great interest in atomic physics, quantum optics, and condensed matter quantum information systems. The principle of the model is analogous to the problem of two coupled classical pendulums that are connected to produce an interaction. This interaction leads to a coupling of the two modes in a way in which all of the energy oscillates between the two. At a given instant of time, pendulum 1 may be static while

all the energy is in the oscillation of pendulum 2, and at a later time the process is in the opposite state. In the quantum case, one pendulum is the harmonic oscillator mode of the electromagnetic cavity, while the second is the atom, which is described as an oscillator by its Rabi frequency. Hence, the field interacts with the atomic state and leads to Rabi oscillations of the atomic state population.

More recently, it was observed that the photon parity operator also shows similar behavior with one important difference. The peaks of the amplitude of the parity operator seem to occur when the Rabi oscillations are at their minimum. So, the amplitude and decay of this parity operator seem to be shifted by half a period from the Rabi oscillations.¹⁰¹ In Fig. 2, we plot the Wigner function of the parity operator at a time when this operator is near its maximum amplitude and at a positive peak (of the oscillating operator). The axes x (real) and y (imaginary) correspond to position and momentum. While the two major peaks are evident, there are clear interference fringes between them. These fringes themselves oscillate at the Rabi frequency when the parity amplitude is large. Because the parity operator is proportional to the number density of the photon field, this latter is also in agreement with the two-pendulum concept, as the Rabi oscillation is a minimum when the majority of the energy is in the photon oscillator.

Many variations on the Jaynes-Cummings model appeared over time. First, the model was generalized through the development of a Markovian master equation approach, rather than sticking with the Hamiltonian form.¹⁰² Then, the model was extended to a double Jaynes-Cummings model by incorporating two two-level atoms within the structure,^{103,104} as well as to a pair of three-level atoms.¹⁰⁵ An anti-Jaynes-Cummings model was devised as well.¹⁰⁶ By reconstructing the Wigner function, these latter authors were able to reconstruct the wave fields themselves. The Jaynes-Cummings model was studied in the strong coupling regime (very small detuning) as a method of investigating broken inversion symmetry.¹⁰⁷ Miranowicz *et al.* replaced the electromagnetic

field with the phonon field in the study of a nanomechanical resonator¹⁰⁸ (for an excellent review on microwave photonics with superconducting quantum circuits, also discussing tomography of microwave photons to reconstruct the Wigner function, see Ref. 109). In Ref. 110, the authors used a Jaynes-Cummings Hamiltonian to analyze a resonant single-atom two-photon quantum optical amplifier both dynamically and thermo-dynamically. It was shown that investigating the Wigner function one can experimentally discriminate between initial field states and even the excitation mechanism (single vs two photon) at long times. In Ref. 111, the authors discussed the Jaynes-Cummings model within the context of Josephson photonics. The quantum dynamics of two electromagnetic oscillators coupled to a voltage-biased Josephson junction was investigated.¹¹²

B. Squeezed states

One of the most ubiquitous objects in quantum mechanics is the Gaussian wave packet, because it can be used to create the minimum uncertainty wave packet. In quantum optics, the minimum uncertainty wave packet is used to represent a single photon, by adding a propagator to a Gaussian wave packet a coherent state can be created

$$\Psi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{4\sigma^2} + \frac{ip_0x}{\hbar}}, \quad (12)$$

which is a propagating minimum uncertainty wave packet. From this coherent state, a Wigner function in phase space can be constructed which moves with the average momentum of the wave packet

$$f_W(x, k) = \frac{1}{\hbar} e^{-\frac{x^2}{2\sigma^2} - 2\sigma^2(k-k_0)^2}. \quad (13)$$

This Wigner function is positive semi-definite; it can be zero at infinite distances from the center in either position or momentum. Now, the idea of a squeezed state is to reduce the uncertainty in either position or momentum of the wave by utilizing nonlinear optics.^{113,114}

In recent work, a squeezed state was used to do phase estimation on an interferometer via the Wigner function for the squeezed state.¹¹⁵ The phase estimation is carried out by using parity detection via the probe state. In recent years, there has been a growing interest in this area, both for better creation of squeezed states as well as for applications of these states. A more general treatment of the role of nonlinearity in creating squeezed states was formulated¹¹⁶ as well as a new time evolution operator to describe the propagation of the squeezed state¹¹⁷ (in Ref. 118, the authors discussed the effect of partially coherent, but decentered annular beams, on the skewness and sharpness of the wave propagation). It was demonstrated that the squeezed state can be generated by a parametric amplifier process,¹¹⁹ in which the squeezed state was generated from a normal harmonic oscillator mode¹²⁰ and from two modes of an optomechanical system.¹²¹ Very recently, two-phonon interactions between mechanical resonators and spin qubits have been studied to

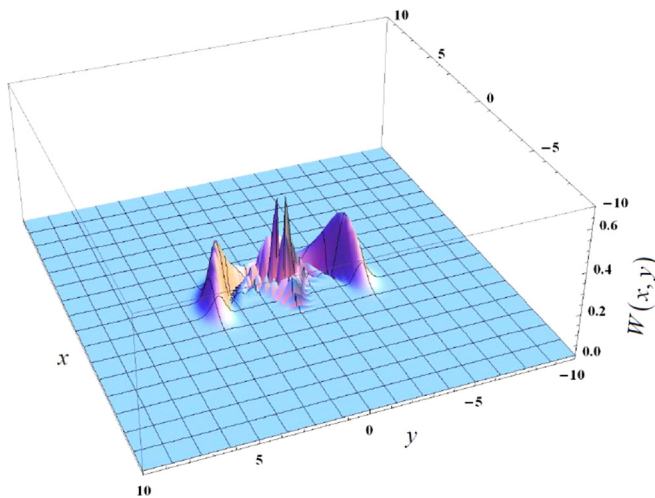


FIG. 2. The Wigner function for the expectation of the parity operator at the scaled time for which this quantity is a maximum (near one-half the revival time). Reprinted with permission from Birrittella *et al.*, Opt. Commun. **354**, 286–290 (2015). Copyright 2015 Elsevier.

show the non-classical states in the presence of dissipation.¹²² Figure 3 shows the dynamics of the density matrix and of a single quantum trajectory as well as the *cattiness* versus time for an initial vacuum state for different parameters.

In addition, the concept of fractional squeezing was introduced.¹²³ The use of a squeezed state to reduce the decoherence of a coherent optical pulse was also demonstrated,¹²⁴ although one might argue that this was the reason squeezed states were developed in the first place. The difference between the quantum nature and the classical nature of the optical pulse was discussed by using a distance measure to determine the difference between the Wigner and the Husimi distribution, and this was claimed to represent a roughness measure for the propagation of the squeezed state.¹²⁵ In Ref. 126, the authors conducted a continuous variable quantum optical simulation for time evolution of quantum harmonic oscillators. By experimentally determining the transient behavior of a quantum harmonic oscillator in an open system and reconstructing the Wigner function, they were able to analytically simulate the dynamics of an atomic ensemble in the collective spontaneous emission by mapping an atomic ensemble onto a quantum harmonic oscillator as an example. Squeezed states and Wigner functions were used for establishing sufficient conditions for the efficient classical simulation of quantum optics experiments that involve inputting states to a quantum process and making measurements at the output.¹²⁷ In Ref. 128, the authors propose a projection synthesis scheme for generating Hermite polynomial excited squeezed—non-Gaussian—vacuum states (the presence of non-Gaussian states is taken as an indication of non-classical behavior). Marshall *et al.* used squeezing operations for a continuous-variable protocol for quantum

computing on encrypted variables.¹²⁹ An excellent overview of squeezed states and their application to laser interferometers is provided in Ref. 130.

In turn, Ref. 131 shows the use in quantum metrology and its application in biology. Finally, a theory of higher-order stochastic differential equations equivalent to arbitrary multi-variate partial differential equations was derived.¹³²

C. Other photon states

The use of specially created photon states, such as squeezed states, has been of considerable interest in the world of quantum information in the optical sciences,¹³³ although other states have been devised as well. The idea of optical vortex states (a state with angular momentum)¹³⁴ was extended so as to produce a squeezed optical state that is also a vortex state.¹³⁵ In other work, the vortex state was created by a nonlocal photon subtraction process.^{69,70} In this latter work, the authors created a photonic chip to continuously generate and manipulate entangled states of light. The entanglement is created by nonlocal photon absorption of two separable states by means of directional couplers with high transmittivity. The resulting delocalized photon is then manipulated by a reconfigurable interferometer, which produces the desired state after photon counting. The final quantum state has the appeal of being both squeezed light and a single photon. In Ref. 71, the authors investigated optical beams with phase singularities (vortices) and experimentally verified that these beams, although classical, have properties of two-mode entanglement in quantum states.

Multimode photon-addition and subtraction of photons arising from Gaussian states can lead to non-Gaussian states, such as the previously discussed squeezed states, which are

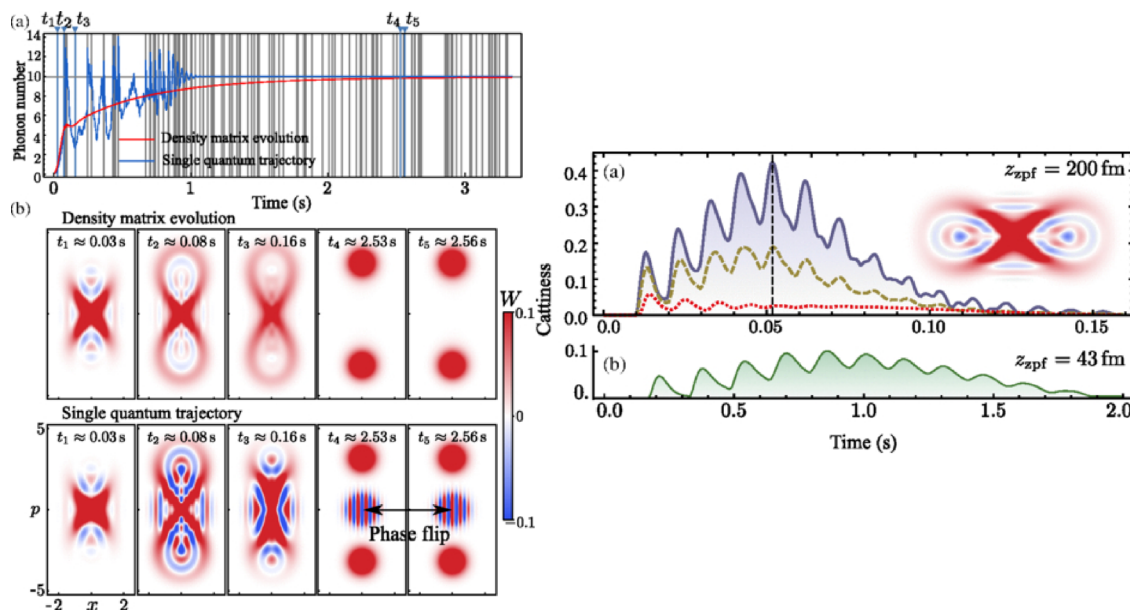


FIG. 3. Left: Dynamics of the density matrix and of a single quantum trajectory. (a) Phonon population versus time. Vertical lines depict times when a phonon emission process takes place. (b) Wigner function of the mechanical mode given by the density matrix (upper) and the wave function of a single trajectory (lower), at the times marked with arrows on the top of panel (a). The last two columns of the bottom row depict a *jumping cat*. Right: *Cattiness* versus time for an initial vacuum state for different parameters. Inset shows Wigner function at the time of maximum *cattiness*. Reprinted with permission from Muñoz *et al.*, Phys. Rev. Lett. **121**, 123604 (2018). Copyright 2018 American Physical Society.

then useful in quantum computation.^{136,137} It was shown that a theoretical framework for such multimode photon-added and -subtracted states can be developed which yields a general Wigner function for the states. Quite generally, quantum computing can outperform classical machines only when entanglement, and the presence of negative components in the Wigner function, exists.⁷² The generation of such a state by homodyne detection, which is basically a subtractive process, was shown to give such a Wigner function.¹³⁸ Such a reconstructed Wigner function for photons at 1.55 μm is illustrated in Fig. 4. The degree of non-classicality in such added and subtracted states used to create squeezed states was characterized recently as well.^{139,140} In Ref. 141, non-Gaussian states, with negative-valued Wigner functions (negativity is related to classical non-simulability), were used to investigate quantum annealing, which is aiming at solving combinatorial optimization problems mapped to Ising interactions between quantum spins.

In cavity optomechanics, light circulating inside an optical resonator is used to manipulate and measure the motion of a mechanical element via the radiation-pressure interaction. After an optomechanical interaction, performing a measurement on the light can be used to conditionally prepare mechanical states of motion. Recently, it was shown that by exploiting the nonlinearity inherent in the radiation-pressure interaction, nonlinear measurements of thermo-mechanical motion in an optomechanical system could be performed.¹⁴² Utilizing the measurement of the displacement-squared motion, the first measurement-based state preparation of mechanical non-Gaussian states was demonstrated. In follow up work, the preparation of quantum superposition states of motion of a mechanical resonator was realized by exploiting the nonlinearity of multi-photon quantum measurements: Initially, classical mechanical interference fringes were observed¹⁴³ followed by the observation of quantum interference¹⁴⁴ (cf. Fig. 5). Other works exponentially enhanced the single-photon opto-mechanical coupling strength by using only additional linear resources¹⁴⁵ and investigated the photon-phonon-photon transfer in optomechanical systems.¹⁴⁶ In Ref. 147, a measurement-based conditional generation of the superposition of mesoscopic states of a nanomechanical

resonator was proposed. The decoherence processes of the generated states are highlighted using the Wigner function. In related work, Van der Pol oscillators were investigated, in particular, concerning quantum synchronization dynamics¹⁴⁸ and oscillation collapse.¹⁴⁹ In Ref. 150, microwave-frequency pulse shaping and conversion between adjustable frequencies by embedding a mechanical oscillator in a tunable circuit were used to demonstrate a temporal and spectral mode converter. The authors measured the Wigner-Ville distribution of the input and the converted output signal to analyze the characteristics of the mode conversion.

D. Detection of photon states

A squeezed state can indeed be detected, usually via reconstructing the Wigner function. One recent approach used point-by-point sampling of the Wigner function via a ultrafast parametric down-conversion process.¹⁵¹ A loss-tolerant time-multiplexed detector, based upon a fiber-optical cavity and a pair of photon-number-resolving avalanche photodiodes, was used. By proper data processing and pattern tomography, the properties of the light states could be determined with outstanding accuracy. Another approach used parity and phase detection within an interferometer to study the two-mode squeezed state and its resulting Wigner function.¹⁵² As may be guessed, the use of quantum tomography is a relatively standard approach to reconstructing the Wigner function, and this is often done with homodyne detection.¹⁵³ This typically includes a phase-sensitive amplifier to amplify the quadrature component of the light. Recently, it was shown that a travelling wave parametric amplifier based upon an optical nonlinear crystal can be used to enhance the quality of the Wigner function reconstruction.¹⁵⁴ Here, the Wigner function contains a negative-valued area along one quadrature and is squeezed along the other. In Ref. 155, the authors showed experimentally the full reconstruction of the polarization quasiprobability distribution from measurements using photon-number resolving detectors. Quantum tomography for reconstructing the Wigner function was also frequently used within the realm of electron quantum optics.¹⁵⁶⁻¹⁵⁹

Other approaches to detecting and observing non-classical states have appeared as well in recent years. For instance, a Knill-Laflamme-Milburn type of interferometer was used to create non-classical states as measured by the negativity of the Wigner function.¹⁶⁰ In other work, a new experimental method for determining non-Gaussian multi-photon states was proposed.¹⁶¹ They recognize that this is a property of Fock states, which are quantum states for a fixed number of particles (see Ref. 162 for an investigation on stabilizing Fock states via Josephson junctions). The authors showed that their proposed approach will give an experimental method of creating the Fock states from normal multi-photon states and provide the increased negativity of the Wigner function.

Aside from detecting photon states, an important challenge is quantifying the non-classicality of these states. Recently, new quantifiers for processing multiple estimates of single-photon state statistics were proposed.¹⁶³ The

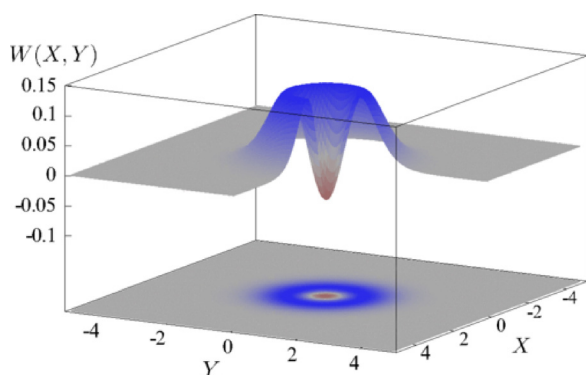


FIG. 4. Reconstructed Wigner function of the single-photon-subtracted, phase-randomized weakly squeezed vacuum state. Reprinted with permission from Baune *et al.*, Phys. Rev. A **95**, 061802 (2017). Copyright 2017 American Physical Society.

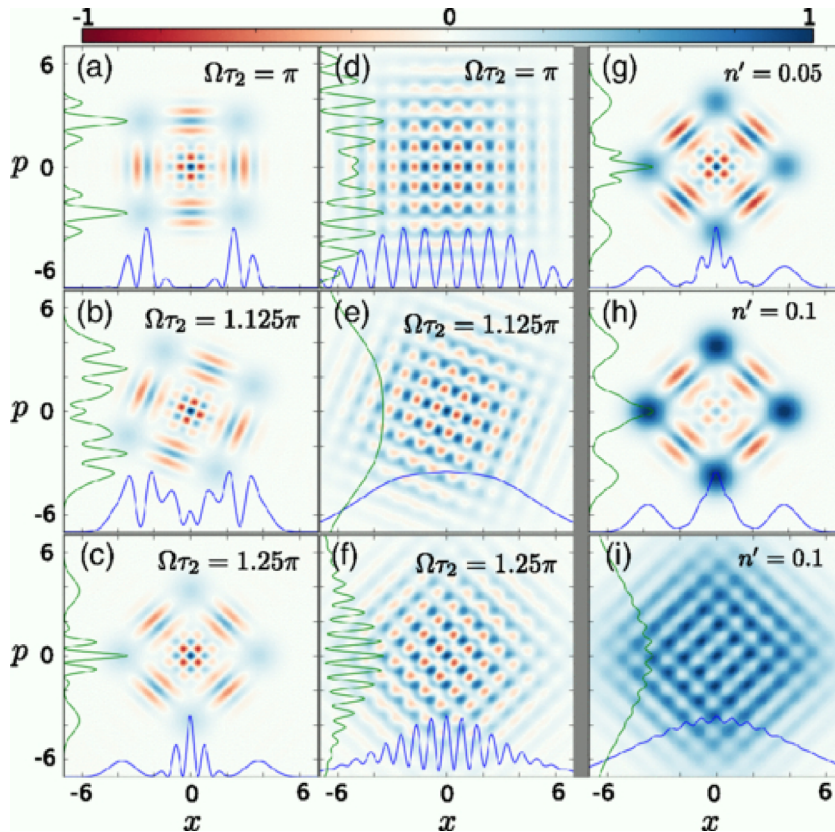


FIG. 5. Wigner distributions of a mechanical resonator's state at different times: The initial *compass* state if the resonator is in the ground state (a)–(c); loss of interference due to initial thermal phonon occupation (d)–(f); during the evolution, the distribution rotates so that the interference fringes oscillate between the position and momentum marginals. Reprinted with permission from Khosla *et al.*, Phys. Rev. X **8**, 021052 (2018). Copyright 2018 Author(s), licensed under the Creative Commons Attribution 4.0 International License.

quantifiers simulate basic capability of single photons to conditionally bunch into a single mode and form a Fock state. This state exhibits complex non-classical phase-space interference effects making its Wigner function negative in multiple areas.

Additional measurement works investigated coarse-graining the phase space of N qubits,¹⁶⁴ continuous variable quantum systems,¹⁶⁵ optically controlled Kerr non-linearity,¹⁶⁶ twisted photons,¹⁶⁷ the potential advantages of heterodyne over homodyne detection,¹⁶⁸ symmetric two-mode squeezed thermal states,¹⁶⁹ a temporal and spatial multiplexed infrared single-photon counter,¹⁷⁰ and probabilistic cloning of arbitrary coherent states.¹⁷¹

E. Schrödinger Cat states again

Schrödinger *Cat* states were defined above as quantum superpositions of classical distinguishable states and allow to generate non-classical states of light as well as to investigate the decoherence process arising during the transition from quantum to classical physics⁸⁸ (cf. Fig. 6). A better understanding of decoherence processes, in turn, allows a grasp of the limits of quantum physical approaches, such as in quantum computing and quantum communications. *Small Cat* states (i.e., having small amplitudes) are referred to as *Kitten* states.

Recently, the optical Schrödinger Cat states emerging in the double-photon pumping and absorbing processes with single-photon decay in one- and two-coupled-modes systems were investigated.⁷³ It was shown that the decoherence effects result in diminishing the negative interference fringes

in the Wigner functions and also suppresses the entanglement and mutual information. In Ref. 172, the authors generated macroscopic Schrödinger Cat states in a diamond mechanical resonator. Here, the Wigner function was used to show the interference fringes and non-classicality of the mechanical Cat state. In Ref. 173, a quantum computing approach was proposed based on Cat states (instead of qubits). The Wigner function was computed based on the numerical solution of the Schrödinger equation. The peaks of the Wigner function are at the classical stable fixed points, and the interference fringes in between denote that two oscillation states are superimposed, i.e., a Cat state. An exact analytical solution of the steady-state density matrix for driven-dissipative systems, including one-photon losses, which are considered detrimental for the achievement of Cat states, was presented.¹⁷⁴ Again, the Wigner function was

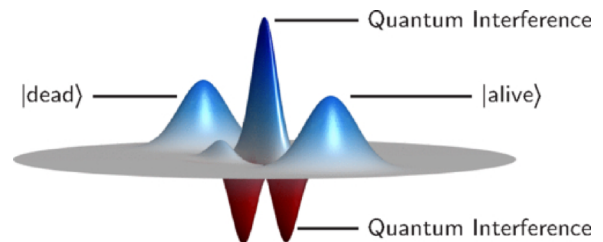


FIG. 6. A Wigner function for a Schrödinger Cat state: The presence of the interference terms indicates that this Wigner function represents a superposition, i.e., it is in both states *alive* and *dead* at the same time. Reprinted with permission from Rundle *et al.*, Phys. Rev. A **96**, 022117 (2017). Copyright 2017 Author(s), licensed under the Creative Commons Attribution 3.0 License.

used to visualize the non-classicality of a state. In Ref. 175, the authors investigated the effect of varying monitoring strategies via quantum measurements to control chaos. The Wigner function was used to analyze the interference generated by the nonlinear dynamics and the way different monitoring strategies (decoherence processes) destroy them. In Ref. 176, a method for generating any bi-photon superposition state using only linear optics was introduced, using a calculated Wigner function to demonstrate non-classical characteristics. Sychev *et al.* demonstrated a conversion of a pair of squeezed Schrödinger Cat states into a single positive Cat state with a larger amplitude.¹⁷⁷ Electronic Cat states were investigated regarding elastic scattering by atomic targets, in particular, the interference effects and the contribution of the Wigner function's negative values.¹⁷⁸

F. Optical qubits

In recent years, the quantum bit, or short qubit, has become of great interest due to the overall interest in quantum computing. A qubit is the quantum analog of the classical binary bit and refers to any quantum system with two levels, which is denoted by $|0\rangle$ and $|1\rangle$. One approach to represent optical qubits is rooted in a qubit-oscillator system approach based on the Jaynes-Cummings model in which the two-state atom is replaced by the qubit. This was recently studied within the rotating wave approximation.^{179,180} This latter work used the results to examine the differences among various information theoretic measures. In particular, the negativity associated with the calculated Wigner distribution was used for indicating the non-classicality of the developing state. Also, as previously hinted, the qubit-oscillator system was studied for broken inversion symmetry.¹⁰⁷

Another approach to represent optical qubits is based on squeezed states. Recently, two-mode qubit-like entangled squeezed states were compared with entangled coherent states, through the use of Wigner functions for the two types of states.⁷⁴ The Wigner function and its negative excursions were used to signify the entanglement of as many as 3000 atoms by a single photon.⁷⁵ The role of entanglement for the generation of the negative portions of the Wigner function was also studied for multi-qubit GHz-squeezed states.⁷⁶ A qubit-oscillator system was used to create Schrödinger Kitten states.^{179–181} By passing these states through a Kerr medium, the gain by such a process can exceed the normal optical losses and lead to squeezed Kitten states.

In other work, a scheme for generating squeezed states of a confined light field strongly coupled to a two-level system, or qubit, in the dispersive regime was proposed.¹⁸² Müller *et al.* investigated cloning of binary coherent states based on the previous work on binary qubit states, using the Wigner function to analyze the statistical moments.¹⁸³

Qubits are especially of interest to applications in quantum computing. Based on the results of Howard *et al.*,¹⁸⁴ Delfosse *et al.* described a quantum computation scheme by state injection on rebits, i.e., states with real density matrices, by using contextuality and Wigner function negativity as computational resources.¹⁸⁵ This was later extended by the

authors by describing schemes of quantum computation with magic states on qubits.¹⁸⁶

G. Bell states

Within the realm of entangled states, a particularly important concept is a Bell state (also frequently referred to as EPR state), which is a maximally entangled quantum state of two qubits. The idea of entangled states in quantum mechanics arises as early as EPR⁸⁵ and the subsequent introduction of the phrase *entanglement* by Schrödinger.⁸⁶ The creation of Bell states became more interesting after the publication of the Bell inequality¹⁸⁷ by Bell in 1964, although the inequality was known already more than a century earlier.¹⁸⁸ Since the inequality was known well before Bell, it can hardly be used to distinguish between classical and quantum mechanics. Nevertheless, Bell states are used to describe entangled states which are necessary for the world of quantum information. At the same time, we point out that there is little to differ between a Bell state and a Cat state.

Recently, entangled non-Gaussian states, obtained from Gaussian entangled states by photon subtraction, were used for continuous-variable quantum information protocols, specifically quantum teleportation.¹⁸⁹ The authors described the teleportation of non-Gaussian, non-classical Schrödinger Cat states of light using two-mode squeezed vacuum light that is made non-Gaussian via subtraction of a photon from each of the two modes. Here, the Wigner function's maximum negativity at the output was used to show how the non-Gaussian entangled resource lowers the requirements on the amount of necessary squeezing. In related work, in Ref. 77, the authors proposed a deterministic scheme for teleporting an unknown qubit state through continuous-variable entangled states in superconducting circuits (we return to this below); the Wigner function was used to measure the amount of entanglement of the entangled coherent state.

In optical communications for quantum information processing, photonic qubits must be processed in complex interferometric networks. This requires synchronization of the arrival times when flying through various media. Recently, it became possible to control the interference between two nearly pure photons that emerge from two independent quantum memories.¹⁹⁰ Controlled storage times of 1.8 μs were achieved with sufficient purity that the Wigner function showed sufficient negative excursions that could be confirmed with homodyne detection.

In other work, control of the entanglement dynamics was achieved via a Kerr nonlinearity that is mediated by cavity detuning in a two-photon process.⁷⁸ Finally, the typical EPR or Bell state was studied with pseudo-spin type of measurements.¹⁹¹ In Ref. 79, the authors investigated the generation and characterization of discrete spatial entanglement in multimode nonlinear waveguides. Pfaff *et al.* experimentally showed the controlled release of multiphoton quantum states (e.g., Schrödinger Cat states) from a microwave cavity memory into propagating modes.⁸⁰ The process allows to generate entanglement between the cavity and the travelling mode.

IV. QUANTUM PHYSICS

The use of the Wigner function is deeply rooted in physics, with some saying it is a third route to achieve quantization.^{21,26} The Wigner function was and is particularly attractive for investigating decoherence processes, i.e., to shine light on the transition of quantum systems from the quantum to the classical regime.^{192,193} For example, decoherence in a one-dimensional attractive Bose gas, due to the presence of random walks in momentum space, was studied.¹⁹⁴ Other work looked at the quantum-classical transition in open quantum dots.¹⁹⁵

The Wigner function plays a key role today in many different areas of physics, such as in investigating quantum systems,^{196,197} various imaging elements in a scanning transmission electron microscope,¹⁹⁸ X-ray investigations of the structure and dynamics of polyatomic molecules,¹⁹⁹ the tomography of multispin quantum states,²⁰⁰ bulk-surface oscillations in driven topological insulators,²⁰¹ measuring complete and continuous Wigner functions of a two-level cesium atom,²⁰² macrospin dynamics in antiferromagnets,²⁰³ and quantum cascade lasers,^{204,205} just to name a few areas. Importantly, in Ref. 67, the authors demonstrated a simple method for quantum state reconstruction that extends those previously known for quantum optical systems to other classes of systems. Using IBM's Quantum Experience five-qubit quantum processor, they showed the reconstruction of two Bell states and the five-qubit GHz spin Schrödinger Cat state (cf. Fig. 7) via spin Wigner function measurements.

Based on this initial outlook on the versatility of the use of the Wigner function in physics, in the following we take a closer look at a couple of key areas and shed some light on the details of the individual application.

A. Harmonic oscillator

The harmonic oscillator is ubiquitous in quantum physics. Groenewold was among the first to treat the harmonic oscillator in terms of the Wigner function.²⁰ It represents one of the few exactly solvable systems using the Schrödinger equation. The normal harmonic oscillator is described by the Hamiltonian²⁰⁶

$$H(t) = \frac{p^2}{2m} + \frac{k(t)}{2}x^2, \quad (14)$$

where p is the normal momentum operator and $k(t) = m\omega^2$.

The harmonic oscillator is used in various forms to describe the quantization of a variety of different fields, whether of electromagnetic origin or of mechanical motion, such as phonons in condensed matter. In these applications, the field is expanded in a number of modes, each of which is described by a harmonic oscillator. Among the areas which apply this is the previously discussed Jaynes-Cummings model (cf. Sec. III A). The description of the field as an array of harmonic oscillators means that it has become natural to describe the background *bath* as an ensemble of harmonic oscillators, and this can be used to study the interaction of other quantum systems with such a bath.

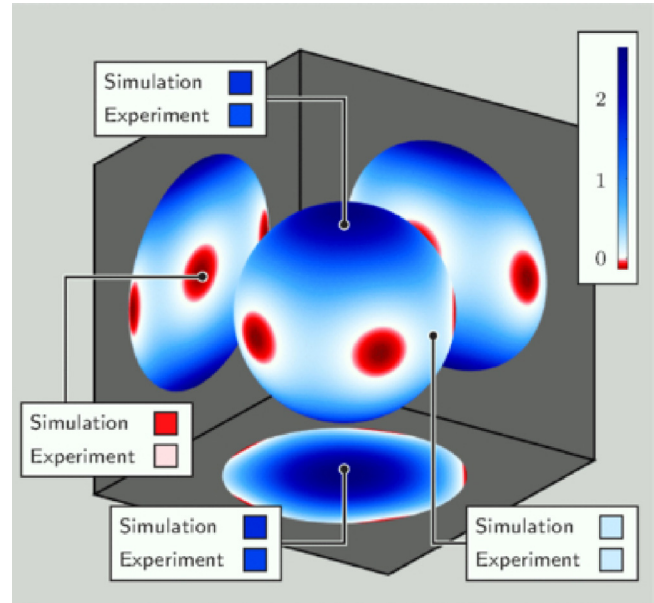


FIG. 7. The five-qubit GHz, spin Schrödinger Cat state Wigner function: The ideal function, and as insets, both simulated and experimental results from IBM's Quantum Experience project are shown. Reprinted with permission from Rundle *et al.*, Phys. Rev. A **96**, 022117 (2017). Copyright 2017 Author(s), licensed under the Creative Commons Attribution 3.0 License.

Recently, the use of a phase space was re-emphasized by a treatment of the driven quantum harmonic oscillator.²⁰⁷ It was shown that the presence of the driving electric field distorts the Gaussian correlation functions that arise in treating the Wigner functions of the oscillator.²⁰⁸ The harmonic oscillator coupled to a heat bath was studied for the case in which the external force is a *kicked* impulsive force.²⁰⁹

In other work, the behavior under decoherence of four different measures of the distance between quantum states and classical states for the harmonic oscillator coupled to a linear Markovian bath was investigated.²¹⁰ In Ref. 211, the authors used harmonic oscillators to describe a state-projected centroid molecular dynamics formalism.

One implementation of the harmonic oscillator uses trapped ion systems. In Ref. 212, the authors observed the quantum interference between two well-separated trapped-ion mechanical oscillator wave packets. The superposed state was created from a spin-motion entangled state using a heralded measurement. Wave packet interference was observed through the energy eigenstate populations. The Wigner function of these states was reconstructed by introducing probe Hamiltonians which measure Fock state populations in displaced and squeezed bases. Also using a trapped ion system, strong nonlinear coupling between harmonic oscillators was achieved by exploiting the Coulomb interaction between two of the trapped ions.²¹³ In Fig. 8, the Wigner functions for different quantum states are shown, both for ones found from the experimental data and the calculated Wigner functions. A trapped ion system was also used to experimentally demonstrate (near) deterministic addition and subtraction of a bosonic particle, in particular, a phonon of ionic motion in a harmonic potential.²¹⁴

More recently, the characteristics of non-Gaussian pure states generated from an anharmonic oscillator, and

approximations to these states created using a truncated Wigner approximation, was studied.²¹⁵ The truncation method removes the third and higher order derivatives that arise from the expansion of the Wigner potential. Since this modifies the corresponding terms in the resultant Liouville equation, or the Wigner equation of motion, there is no guarantee that any non-Gaussian correlations predicted by the truncated Wigner function would be accurate. However, these authors showed that such non-Gaussian states can be reliably predicted, even with the truncation mentioned. The truncated Wigner approximation is used in many other areas as well, such as in analyzing conservation laws and dynamics of a one-dimensional Bose gas,²¹⁶ the exciton transport in open quantum systems,²¹⁷ many-body localization and thermalization using semiclassical methods,²¹⁸ and non-equilibrium dynamics of spin-boson models from phase-space methods.²¹⁹

Very recently, the Wigner function was used to investigate the transition to bistability in a quantum nanomechanical oscillator.²²⁰ A classical oscillator was capacitively coupled to a quantum dot and comparisons of the response with the harmonic oscillator were drawn. The Wigner function and the harmonic oscillator play an interesting role in nuclear physics, more precisely in a concept termed the *scissors* mode, which is a counter revolution of the protons against the neutrons in deformed nuclei.²²¹

B. Quantum Hall effect

The study of the quantum Hall effect has already yielded two Nobel prizes, i.e., for the effect itself and for the fractional quantum Hall effect. To this day, there remains significant interesting physics in these systems and Wigner functions play a key role to unravel them. Recently, the Levitons were examined as excitations in the fractional quantum Hall effect²²² and a form of crystallization of these excitations was shown to occur. Such excitations were studied in two-dimensional topological insulators where three-electron collisions can be found.¹⁵⁷ These excitations were also demonstrated to have a use in two-particle

interferometry in the quantum edge channels, a process which may have some impact in some approaches to quantum computing.¹⁵⁸

C. Superconductivity

Wigner functions find their application within the general realm of superconductivity, in particular, for studying Josephson junctions and qubits. The Josephson junction is a tunnel junction in which the materials on either side of the insulator (or non-superconducting material) are superconductors. Josephson junctions have been previously mentioned in Sec. III, in particular, concerning Josephson photonics: In Ref. 112, the authors coupled two electromagnetic cavities to a Josephson junction. Here, the circuit is a series connection of the two cavities with the junction. A variation on the above concept is using two cavities which are coupled by a single Josephson junction.¹¹¹ In this situation, it is observed that the tunneling of a Cooper pair through the Josephson junction, from one cavity to the other, excites two photons, one in each of the cavities, or resonators. These microwave photons leak out and can be observed experimentally. The role of nonlinearities, arising from the nonlinear current through the junction, can be important in the system. The coupling parameter is determined by the effective parameters of the individual oscillators. If the coupling to both loops is reduced to zero, one gets a non-degenerate parametric amplifier, while if it is coupled to only a single resonator, the authors suggest that they achieve an anti-Jaynes-Cummings model, in that the driving field de-excites both the resonator and the junction. In the latter case, the Wigner function demonstrates negative regions of correlation between the resonator and the junction.

A Josephson junction coupled to a superconducting ring creates two states that can be monitored and can be entangled, allowing to create qubits from such a structure. One approach was to use superconducting quantum interference devices (SQUIDs) in which a small superconducting island coupled via two ultrasmall, but identical, Josephson junctions.^{223,224} They worked in the charge state basis, in

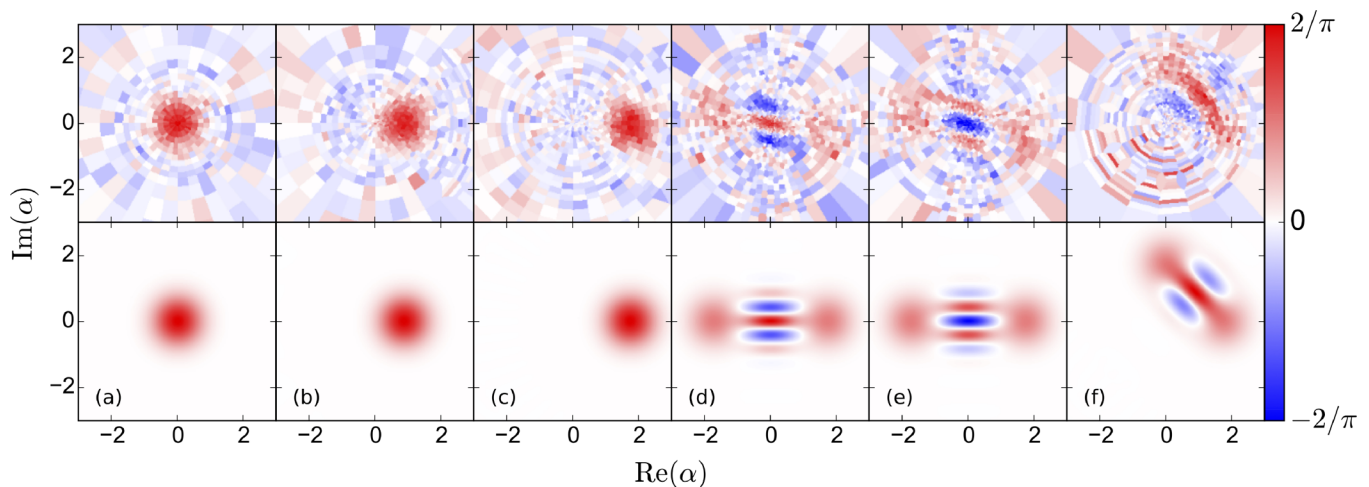


FIG. 8. Wigner functions for different quantum states. (a)–(f) Wigner functions (from left to right) for the vacuum state, and coherent as well as Schrödinger Cat states for different configurations. The top row corresponds to the experimental data, while the bottom row shows the calculated Wigner functions. Reprinted with permission from Ding *et al.*, Phys. Rev. Lett. **119**, 150404 (2017). Copyright 2017 American Physical Society.

which the single-electron charging energy was much larger than the Josephson coupling constant. Then, the state of the qubit can be read by a single-electron transistor capacitively coupled to the charge qubit.

Dissipation in these Josephson structures will of course destroy the entanglement and the corresponding negativity of the Wigner function,²²⁵ but it appears in different forms here. It was demonstrated that a variety of states can be created in these Josephson systems, including the Schrödinger Cat state, squeezed states, and entangled states.¹⁸⁰ Furthermore, it was shown that a system with dissipation can be stabilized by applying an active feedback from an ancillary system, or by reservoir engineering.^{78,226} The study of these systems was also extended to an array of qubits in an electron ensemble.²²⁷ Finally, the reader is directed to an excellent recent review of the coupled microwave resonator and the superconducting qubit.¹⁰⁹

D. Plasmas

The Wigner function entered the stage for plasma physics around 1991 with the effort to establish an equation of motion for the Wigner function.²²⁸ Since then it played a key role in this field. Recently, a Feynman path integral approach to finding the Wigner function for a canonical plasma was developed.^{229,230} In Ref. 231, the authors derived the nonlinear kinetic equations for the degenerate Fermi-Dirac distribution in an electron plasma. This was followed by an investigation of the general requirements for the spin-1/2 quantum plasma of electrons.²³² The effects of radiation transfer from a fusion reactor were also studied with the Wigner function.²³³

The quantum Brownian motion of particles in a plasma, in the presence of damping and the diffusion of the particles, was also studied.²³⁴ The authors discussed the cases of linear and quadratic coupling in great detail and derived the stationary solutions of the master equation for a Brownian particle in a harmonic trapping potential using the Wigner function. This work paves the way for studying the properties of impurity atoms embedded in a Bose-Einstein condensate or an ultracold Fermi gas.

A two-dimensional, one component plasma was studied in the cell model with cylindrical symmetry.²³⁵ In this work, the counter ions are placed at the center of the cylindrical space with the electrons in the larger area around them. The role of strong correlations in a two-component, degenerate plasma was also studied recently.²³⁶ To that end, the authors developed a new path integral representation of the Wigner function representing the canonical ensemble.

The Wigner function was also used in determining the hydrogen emission line in a low density hydrogen plasma typical of interstellar situations.²³⁷ Furthermore, a weakly ionized, hydrogen-based plasma was studied for the quantum situation, and the wave dispersion relation and statistical properties were determined with a Wigner distribution approach.²³⁸

E. Relativistic systems

Schrödinger proposed an extension of his equation that meets the requirements of special relativity for spin-less particles, which is better known today as the Klein-Gordon (or

Klein-Gordon-Fock) equation.^{239–242} This was subsequently followed by the more famous Dirac equation for spin particles.²⁴³ Relativity is not only relevant to extremely large scales such as astrophysics but also in the very small scale, i.e., on the molecular, atomic, and sub-atomic level. Relativity and quantum physics—and as we will see by extension the use of the Wigner function—are thus not only relevant but also even important over the entire scale spectrum.^{39,244} Within the realm of particle physics, strong magnetic fields were particularly investigated with the use of the Wigner function, in particular, regarding fermions²⁴⁵ and for spin-1/2 particles.²⁴⁶ In other work, the Wigner formalism was applied to the study of quantum cosmology of the universe, where the concept of deformation quantization was applied.²⁴⁷ The authors studied a Friedman-Lemátre-Robertson-Walker model of the universe that is filled with radiation and dust or cosmic strings. The relativistic equation of motion contains the Wigner function, which is shown in Fig. 9 for a cosmic string filled universe. There is apparently a turning point at $x = 0$ in these coordinates, as one can see from the yellow line for classical behavior. The Wigner function determined from the equation of motion has significant oscillatory behavior that arises both inside and outside the classical trajectory. The relativistic situation for the universe model does not seem to add much to the non-relativistic approach to semiclassical quantization. In other work, the role of the chiral anomaly and vorticity in the condensation of massive fermions was also discussed using the Wigner functions.^{248,249}

In general, the quantum treatment of particles and electromagnetic waves, including relativistic effects, is referred to as quantum electrodynamics. In quantum electrodynamics, the particles are the electron, proton, neutron, and photon (and their anti-particles), where the photon is the carrier of

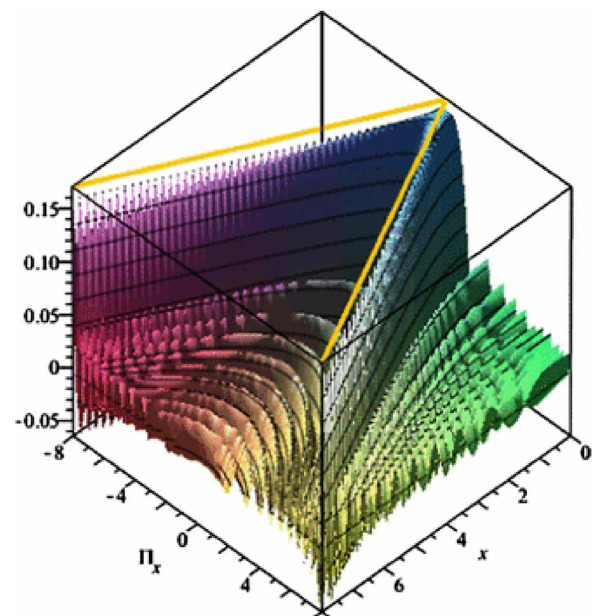


FIG. 9. The Wigner function of a cosmic string filled universe. The corresponding classical trajectory is denoted by the yellow line. Reprinted with permission from M. Rashki and S. Jalalzadeh, *Gen. Relativ. Gravitation* **49**, 14 (2017). Copyright 2017 Springer Nature.

the electromagnetic field and interaction. Some of these particles are charged and some are fermions (electron and proton) and bosons (neutron and photon). The Wigner function has become a key tool in quantum electrodynamics, e.g., recently the magnetic effect in quantum electrodynamics on heavy fermions was investigated.²⁵⁰

Further investigation on how the nucleus is held together yielded the so-called *strong force* as well as quantum chromodynamics and in turn a new set of fundamental particles (e.g., quarks)—a hadron is a composite particle made of quarks held together by the strong force—of which the previous fundamental particles are composed. In Ref. 251, the authors generated a Wigner phase space distribution for the quark transverse position and its momentum in a proton and showed that this is sufficient to evaluate their cross product, the orbital angular momentum. Based on quarks and the strong force carrying particles, gluons, a similarity between quark-gluon interactions and electron-photon interactions was established. Then, the kinetic theory of the quark-gluon plasma can be treated by means of gauge-covariant one-particle Wigner distribution functions for the quarks, antiquarks, and gluons. This similarity was used in many instances, but particularly recently in Ref. 252, where in the quark-gluon plasma, the chiral symmetry is restored to the system, and chirally invariant transport equations could be written in terms of the Wigner function. A coarse-graining of the Wigner function, into a Husimi function, was used to determine the entropy production from relativistic heavy-ion collisions in the case where the dynamics has gone chaotic.²⁵³ The Wigner function was used in the context of electromagnetic fields to describe the behavior of photons²⁵⁴ and fermions.²⁵⁵ Another aspect of the heavy ion collisions is the generation of so-called *parton* jets. Here, the partons are simply the free hadrons that are emitted during the collision event of the two heavy ions. These impact the so-called vacuum state surrounding the colliding ions, and Wigner functions are useful here to describe the nature of the vacuum²⁵⁶ and the parton shower.²⁵⁷

The Wigner function was also used to study the production of quark pairs during the plasma formation,²⁵⁸ as well as the specific role of the bottom quarks (one type of quark).²⁵⁹ Many more applications of the Wigner function within the context of strong force are available.^{260–263}

Aside from applications in investigating the strong force and related challenges in particle physics, gravity is investigated to this day with the use of the Wigner function. In particular, the wave-kinetics of acoustic-gravity waves were studied.²⁶⁴ Also, the effect that quantum gravity has on the normal quantum harmonic oscillator was studied from the approach of the generalized uncertainty principle.²⁶⁵ The differences may or may not be measurable, depending upon the size of the Planck mass. In related work, the effects of noise on the entanglement enhanced gain in interferometer sensitivity were investigated using the Wigner function.²⁶⁶

V. QUANTUM ELECTRONICS

Quantum electronics describes electron-based devices where quantum effects are primary factors. Historically

quantum electronics focused on the interactions of electrons and photons, e.g., quantum cascade light amplification by stimulated emission of radiation (LASER) devices, which is why quantum electronics is often linked to quantum optics. However, quantum electronic devices go way beyond LASER operation: Fundamental is the understanding of quantum electron transport which is the base for many use cases and was investigated in detail in the last decades within the context of electronic devices with Wigner functions playing a key role.^{39,56,57,159,244,267–271}

When modelling or simulating electronic devices, there are several challenges to consider. First and foremost, for simulations are the stability, convergence, and computational effort of the techniques. Also, because a transport system is, by necessity, an open system to allow for contacts: The nature of the boundary conditions is of importance, especially as these may well dominate questions of stability and convergence. Self-consistency between solutions of the Wigner equation of motion and solutions of Poisson's equation is of utmost concern. Finally, the role of collisions, i.e., scattering effects, is exceedingly important as is the inclusion of the electromagnetic field.

In this section, we first discuss different recent Wigner function based numerical and computational aspects followed by approaches to handle scattering and electromagnetic fields. Finally, we highlight recent advances in quantum electron devices.

A. Numerical and computational aspects

A key aspect of solving the Wigner equation of motion (8) is to handle the derivatives in time, position, and momentum. This has been known since the earliest work in devices.^{39,272} A widely applied approach is to use deterministic approaches based on a finite difference scheme. In recent work,²⁷³ the authors suggest to use momentum domain narrowing to reduce the computational complexity. The rationale behind it is that there are regions where one does not need the complete numerical accuracy to resolve the Wigner equation. Also, the steady-state solution of the discrete Wigner equation was investigated.²⁷⁴

Here, the balance between the degrees of numerical accuracy in the evaluation of the kinetic and potential terms that, respectively, describe two actions in orthogonal directions of the phase space was investigated. The effect of the uncertainty principle on the solution of the Wigner equation was analyzed.²⁷⁵ It was shown that the cross-diagonal-position uncertainty of electrons within the device results in a restriction on the minimum momentum resolution of the Wigner function, again something that was known to the early practitioners.⁵³ In Ref. 276, the authors developed a novel deterministic solution method based on a weighted essentially non-oscillatory (WENO) scheme. The approach was compared to a reference non-equilibrium Green's functions²⁷⁷ solution of a resonant tunneling diode.

Among the problems when using a finite difference approach, however, is that the electric field exists on an intermediate spatial grid that is interpolated between the normal grid points. Computation with the nonlocal potential (or

even with the local field) is the major difficulty in the self-consistent solution, as the Wigner potential extends over the entire space of interest, even if the potential itself does not. Based on the work of Ringhofer, who suggested that it may be more efficient to use a spectral method in which Fourier transformation of the potential is introduced,²⁷⁸ there is recently a renewed interest in spectral methods: In Ref. 279, the authors developed a method which uses a spectral decomposition into $L^2(\mathbb{R}^d)$ basis functions in momentum-space to obtain a system of first-order advection-reaction equations. The resulting equations are solved by splitting the reaction and advection steps so as to allow the combination of numerical techniques from quantum mechanics and computational fluid dynamics by identifying the skew-Hermitian reaction matrix as a generator of unitary rotations. Another approach is based on resolving the linear advection in (x, t) -space by an explicit three-step characteristic scheme coupled with a piecewise cubic spline interpolation, while the Chebyshev spectral element method in k -space is adopted for accurate calculation of the nonlocal pseudo-differential term.²⁸⁰ The stability and accuracy of a pseudospectral scheme for the Wigner equation were investigated as well.²⁸¹ In other work, the Wigner-Liouville equation was reformulated using a spectral decomposition of the classical force field instead of the potential energy.²⁸² This simplifies the Wigner-Liouville kernel both conceptually and numerically as the spectral force Wigner-Liouville equation avoids the numerical evaluation of the highly oscillatory Wigner kernel which is nonlocal in both position and momentum. The authors evaluated their approach via a simulation of a resonant tunneling diode. Figure 10 shows the Wigner function near resonance. One clearly sees *negative regions* where the Wigner function takes negative values, reflecting non-classical behavior. These negative regions sustain a current, carried by a jet of positive quasiprobability density electrons that tunnel through the resonant level and are accelerated on the right-hand side of the barriers.

Other recent research on solving the Wigner equation considered a spatially dependent effective mass,^{283,284} which is particularly important for investigating heterostructure devices.²⁸⁵ Heterostructures contain spatial variations of the effective mass, which usually are not considered in regular solution approaches of the Wigner equation. Yet, this can be quite important if one wants to analyze a resonant tunneling diode device.

As previously hinted, boundary conditions are a critical aspect for fully modelling electronic devices.²⁸⁶ In Ref. 287, the authors considered the existence and uniqueness of the solution of the Wigner equation in the presence of boundary conditions. In other work, the drawbacks of using local boundary conditions as well as upwind or hybrid difference schemes with regard to coherence effects were studied.²⁸⁸ Based on this analysis, the authors proposed a nonlocal boundary condition scheme, which includes the effect of the drift operator. Unphysical results leading to oscillations near the boundaries could be prevented. The coherence effect is inherently included and conservation laws were obeyed.

Aside from deterministic approaches, a widely favored method to solve the Wigner equation is to use a stochastic

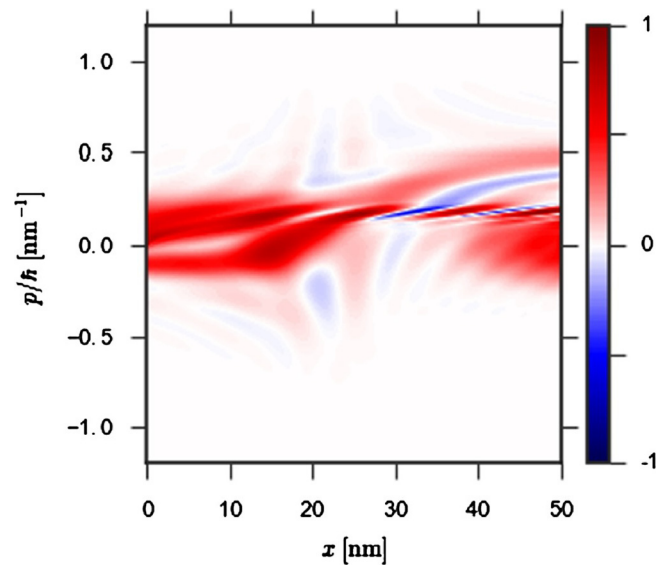


FIG. 10. The Wigner function of a resonant tunneling diode near the resonance peak in arbitrary units. Reprinted with permission from Van de Put *et al.*, J. Comput. Phys. **350**, 314–325 (2017). Copyright 2017 Elsevier.

approach, in particular, kinetic Monte Carlo methods further extended to include concepts of weight, affinity, or signed particles.^{39,289} Regardless of the particular method, among the advantages and disadvantages, one advantage of kinetic Monte Carlo approaches stands out from the computational side: The option is to exchange memory requirements (which is a key show stopper, in particular, for higher dimensional problems) with computing time. This flexibility combined with other features ensured that the Monte Carlo method remained a highly attractive option over the years.^{56,57,290–295}

In recent years, the signed particle approach (the corresponding concept is discussed in Refs. 268 and 270) implemented in the free open source Wigner Ensemble Monte Carlo simulator shipped with *ViennaWD* [<http://www.iue.tuwien.ac.at/software/viennawd/>] (allows to simulate quantum transport of arbitrary one- and two-dimensional problems) has been extended to tackle the computational burden by a parallelization approach, significantly reducing the simulation runtimes.^{296,297} Another signed particle approach was implemented in the *nano-archimedes* [<http://www.nano-archimedes.com/>] software package.

Very recently, a novel numerical solution technique—termed lattice Wigner method—was introduced which solves the Wigner equation based on a lattice discretization of momentum space.²⁹⁸ This approach was derived from the lattice Boltzmann method, originally introduced as an alternative to the discretization of the Navier-Stokes equations of continuum fluid mechanics. This approach is particularly attractive due to the reduction of the momentum space to a comparatively small set of representative momentum vectors, which opens up interesting prospects for the simulation of one-, two-, and also three-dimensional quantum systems.

B. Scattering

Scattering happens when electrons collide with other entities, such as impurities, lattice vibrations (phonons), or

other electrons. Considering scattering effects is increasingly important as scattering causes decoherence which in turn destroys quantum coherence.²⁹⁹ In principal, the Wigner equation offers the ability to include scattering via inclusion into the Hamiltonian. That being said, the general task of including scattering into quantum transport modeling is far from straightforward and remains a challenging task (for a history on including scattering effects into the Wigner equation see, for example, Ref. 270). Recently,³⁰⁰ the signed particle approach for the Wigner transport model was used for analyzing the electron state dynamics in quantum wires focusing on the effect of surface roughness. The limits of local scattering models to describe dissipation and decoherence phenomena were depicted and based on this non-local quantum operators were devised to overcome those limits.³⁰¹ In Ref. 302, the electron transport behavior in transition metal dichalcogenides with a dominant spin-orbit interaction was investigated. The authors showed that this leads to a 2×2 Wigner matrix for either the conduction band or the valence band. The off-diagonal elements display interference phenomena from the two diagonal components which are relevant, for instance, for modelling electron scattering and spin coherence. A critical view on the application of the Boltzmann collision operator for the time-dependent modeling of dissipative quantum transport was discussed for the Wigner picture.^{303,304} The authors discussed unphysical results and suggested to tackle this problem via Bohmian conditional wave functions. In Ref. 305, it was shown that the coherent multiple scattering of electrons in a one-dimensional disordered system leads to the slowdown of its dynamics due to weak localization. A transport model using the Wigner-Rode formalism to investigate thermoelectric properties of periodic quantum structures was developed.³⁰⁶ The approach covers the full electronic bandstructure as well as carrier scattering with phonons and ionized impurities, enabling one to simulate both energy relaxation and quantum effects from periodic potential barriers.

C. Electromagnetic fields

The motion of classical particles is governed by forces, which at any instant act locally causing acceleration over Newtonian trajectories. A charged particle, moving in an electromagnetic medium, experiences the Lorentz force, comprised by the joint action of the electric and magnetic fields. Therefore, for a full electromagnetic transport picture in solids, a full treatment of the electric and the magnetic forces must be conducted. A Wigner approach is uniquely qualified for this task. In a recent work, the Wigner theory for Bloch electrons in homogeneous electric and magnetic fields of arbitrary time dependence was developed.³⁰⁷ The resulting equation for free electrons in constant magnetic and electric fields resembles the classical Boltzmann counterpart. The approach is then generalized for arbitrary energy bands.

Very recently, a general treatment of electromagnetic fields within the Wigner picture was developed based on previous investigations regarding the choice of the gauge³⁰⁸ using the introduction of the Weyl-Stratonovich transform of

the density matrix.³⁰⁹ The Weyl-Stratonovich transform introduces a function of phase space variables corresponding to the kinetic momentum and position. The equation of motion was derived for the case of scalar and vector potentials corresponding to general, inhomogeneous, and time-dependent electromagnetic conditions. The equation is independent of a particular choice for the gauge, which makes it particularly attractive for numerical implementations.

D. Quantum electronic devices

In this section, we show use cases of Wigner function-based simulations of quantum electron devices. The historic preeminent example is the resonant tunneling diode, which has been already mentioned several times. Conceptually, a resonant tunneling diode is composed of a quantum well placed between two potential barriers. The bound state in the well provides *resonant* tunneling through the structure, which in turn provides a filter on the electronic states which pass through, e.g., this structure is the electronic equivalent of an optical resonator. As a result, the device is clearly a quantum mechanical structure and exhibits negative differential conductance in the current, which can be used for a two-terminal oscillator. Historically, the resonant transmission of electron waves through double barriers dates back to 1951 and Bohm⁸⁹ and is used to this day as testbed to investigate quantum transport effects and novel numerical and computational approaches with Wigner function approaches.^{273,274,276,284,310–314} In particular, Ref. 310 deals with recent work on dissipative transport of the resonant tunneling diode and offers quite nice new information on the current and the dissipative parameters. Figure 11 depicts the Wigner distribution function calculated for the bias voltage corresponding to the peak of the current without and with scattering.

Aside from the prototypical quantum mechanical electronic device, the resonant tunneling diode, the Wigner function was also used to model the quantum electron transport in metal-oxide-semiconductor field-effect transistors.^{315–317} There the Wigner function is particularly useful to identify the ballistic and diffusive transport regimes (scattering effects).

Modern day quantum electronics, however, is primarily concerned with confined devices in the nanometer regime, in particular, graphene, quantum wires (also known as nanowires), and quantum dots, representing a two-, one-, and a zero-dimensional transport system, respectively. Those confined quantum systems potentially open paths to sustain the high pace of electronics, be it challenges regarding scaling limitations of conventional electronics (e.g., memory applications³¹⁸), energy generation (e.g., solar cells³¹⁹), energy storage (e.g., memory devices³²⁰), novel computing approaches (e.g., binary atomic silicon logic³²¹), and many more. Two-dimensional ballistic electron quantum transport in graphene has been investigated and similarities with the classical picture have been identified.³²² The results show that the quantum treatment is particularly important for strong electric fields, as the model predicts a non-negligible correction to the charge inside the device (similar to corrections that occur

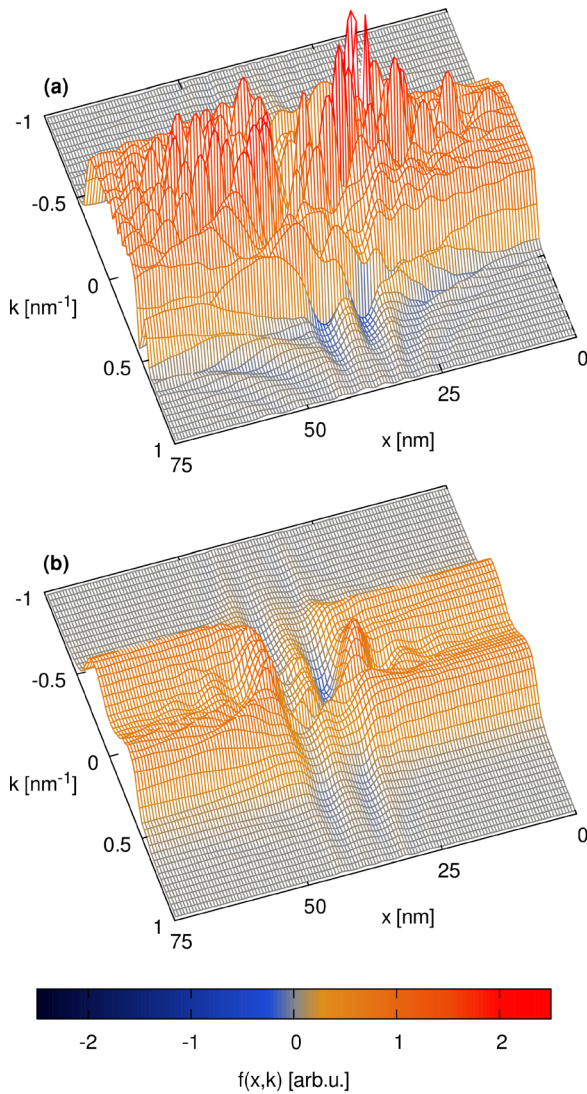


FIG. 11. The Wigner distribution function calculated for the bias voltage corresponding to the peak of the current without scattering (a) and with scattering (b). Reprinted with permission from M. Wołoszyn and B. J. Spisak, Phys. Rev. B **96**, 075440 (2017). Copyright 2017 American Physical Society.

with quantum potentials³²³). Other work dealt with a fluid-dynamic model for electron transport in graphene.³²⁴ Regarding nanowires, based on earlier work on simulating wave guide transport with the use of Wigner functions,^{325,326} where investigations were restricted to elastic scattering by impurities and low lattice temperatures or ballistic transport, modeling capabilities improved over time. The electron-

phonon interaction could be simulated in a quantum wire³²⁷ as well as the quantum transport in a carbon nanotube field-effect transistor³¹⁵ and in a silicon nanowire,³²⁸ just to name a few. More recently, some investigations of quantum wires^{300,329} and quantum dots³³⁰ were conducted within the Wigner picture. Figure 12 shows the Wigner function of a gate-all-around silicon nanowire transistor for different outer radii of the silicon shell.

Wigner functions were also used in modeling quantum well lasers,³³¹ in particular, for modeling the carrier dynamics and the effect of the boundary conditions. More recently, a Wigner function approach was used to simulate partially coherent, dissipative electron transport in biased semiconductor superlattices,²⁰⁴ particularly those for quantum cascade lasers. There a GaAs/AlGaAs superlattice was investigated based on a model collision integral with terms that describe energy dissipation, momentum relaxation, and the decay of spatial coherences.

Other quantum devices, such as quantum ratchets^{332–334} (i.e., periodical structures with broken spatial symmetry; also known as *Brownian motors*) and quantum shuttles³⁰ (i.e., a nanoelectromechanical system that transports electrons one by one by utilizing a combination of electronic and mechanical degrees of freedom) have also been studied with the Wigner function.

Aside from *known* device concepts, the Wigner transport picture also enables to investigate novel device concepts. Recently, a new branch of quantum electronics is materializing coined *entangletronics*, short for entangled electronics.³³⁵ This term describes applications and approaches where the fundamental aspect is the manipulation of the electron dynamics via processes maintaining coherence (here, entanglement is understood to represent the complicated coupling between device and contact states); coherence describes all properties of the correlation between physical quantities of a single wave, or between several waves or wave packets. However, scattering processes strive to counteract coherence and therefore have a strong impact on the entire process. In Ref. 335, the authors presented a two-dimensional analysis of lens-governed Wigner signed particle quantum dynamics to investigate manipulation of electron evolution via electrostatic lenses. Very recently, in follow up work,³³⁶ the electron interference in a double-dopant potential structure was investigated by comparing classical with quantum simulation results: A unique feature of a Wigner function-based transport modeling approach. By

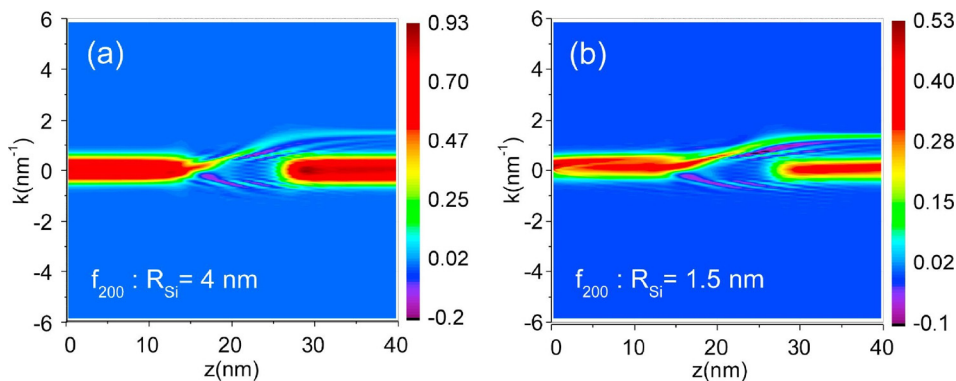


FIG. 12. The calculated Wigner function of an active channel for two different radii (R_{Si}) of the outer silicon shell of a gate-all-around nanowire transistor. Quantum interference patterns and hot-electron relaxations at the drain electrodes (right) are identified. Reprinted with permission from Lee *et al.*, Solid-State Electron. **139**, 101–108 (2018). Copyright 2018 Elsevier.

being able to directly compare the classical with the quantum world, quantum effects could be clearly identified and similarities with the famous double-slit experiments were highlighted. Figure 13 shows the interference pattern of the investigated double-dopant structure surfacing in the quantum case. Although not explicitly associated with entanglement, other recent work dealing with quantum dynamics of wave packets interacting with potential barriers using a Wigner function approach is related.^{337,338}

VI. QUANTUM CHEMISTRY

Quantum chemistry has a long history dating back to Schrödinger, but also to Heitler's and London's milestone work published in 1927 on applying quantum mechanics to the investigation of the diatomic hydrogen molecule and the chemical bond.³³⁹ Since then, quantum chemistry evolved and is a highly active field today³⁴⁰ and in some areas, Wigner functions are a key tool, as we will see in the following.

As hinted above, since the beginning of quantum mechanics, the analysis of molecules and atoms with their spectra was a primary concern. An attractive approach is to find a Wigner function based on path integrals³⁴¹ to describe quantum correlation functions. Based on the groundwork of Poulsen *et al.*³⁴²—who developed a linearized Feynman-Kleinert variational path integral approach to study, e.g., the correlation functions for a chain of He atoms—the Feynman-Kleinert estimation of the density matrix and the resulting linearized path integral were further developed.^{343,344} Here, the approach was applied for determining the dynamic structure factor in parahydrogen and ortho-deuterium. The

Wigner representation of the density operator was also applied to study the correlation functions in the spin-boson model,³⁴⁵ where the authors calculated the equilibrium population difference as a bias was applied.

A key interest in quantum chemistry is the connection between the quantum dynamics and the classical Boltzmann distribution. In Ref. 346, the exact time-correlation functions for the normal modes of a ring-polymer were developed in terms of Feynman paths. Taking the limit of an infinitely long polymer, the authors found that the lowest mode frequencies take their Matsubara frequency values (as determined from, e.g., Matsubara Green's functions). Wigner-Moyal transformations of the correlation functions allow the connection to the classical phase space dynamics. Tanimura³⁴⁷ studied the real and imaginary time (as used in the Matsubara Green's functions) for a hierarchical set of Fokker-Planck equations. These were then used to study the phase space Wigner dynamics of a model quantum system coupled to a harmonic oscillator bath.

Also, a molecular dynamics approach for determining the microcanonical distribution and connecting it to the Wigner function was demonstrated.²¹¹ In Ref. 348, the Talbot-Lau matter wave interferometer was investigated with potential experimental applications in the context of antimatter wave interferometry, including the measurement of the gravitational acceleration of antimatter particles.

The dependence upon the initial condition for evolution of the photodynamics was studied for the pyrrole molecule, with the quantum situation being described via a Wigner distribution.³⁴⁹ It was found that the use of the quantum distribution (using quantum sampling based on a Wigner distribution) already from the initial conditions fits to the evolving photodynamics much better than a classical thermal distribution. The underlying technique of sampling a Wigner distribution was also used in elucidating the photophysical mechanisms in sulfur-substituted nucleobases (thiobases) for designing prospective drugs for photo- and chemotherapeutic applications³⁵⁰ as well as in revealing deactivation pathways hidden in time-resolved photoelectron spectra.³⁵¹

The Wigner function proved particularly attractive for investigating tunneling effects. The use of a Wigner function in the chemical reaction process enables to study the bridge between classical and quantum treatments in various molecular and chemical systems. In Ref. 352, the tunneling-based dissociation of the H atom in electronically excited pyrrole was investigated. In this work, the authors used a trajectory-based approach to calculating the various tunneling probabilities from the phase integrals. The trajectories were limited to straight-line tunneling paths, and it was found that sampling these paths based upon a fixed energy Wigner distribution gave the best fit to the quantum mechanical dissociation rates. Most interestingly, during the study of chemical reactions a conundrum materialized. Some studies showed vanishing tunneling times,^{353–355} or times that are independent of the barrier thickness.³⁵⁶ These results might suggest that non-relativistic quantum mechanics is violated. Recent work³⁵⁷ suggested that one should use the transition time or in other cases a tunneling flight time but shows that their results tend to agree with estimates using a Wigner tunneling packet. In

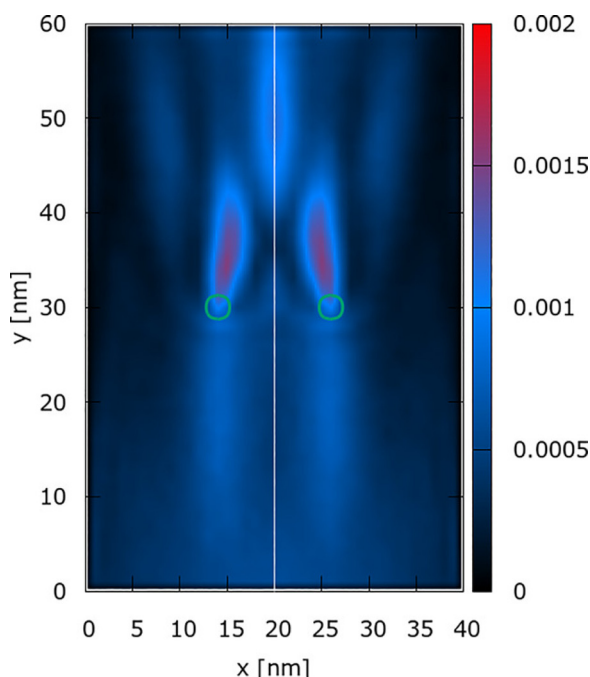


FIG. 13. Quantum electron density ([a.u.]) after 200 fs of the initial minimum uncertainty condition. The green circles are isolines at 0.175 eV of the Coulomb potentials modeling the dopants. Reprinted with permission from Weinbub *et al.*, Phys. Status Solidi RRL **12**, 1800111 (2018). Copyright 2018 Author(s), licensed under the Creative Commons Attribution 4.0 License.

Ref. 358, the authors use the Wigner function to investigate excited-state intramolecular proton transfer for o-nitrophenol where tunneling plays a key role.

VII. SIGNAL PROCESSING

The study of propagation of either acoustic or electromagnetic waves through the environment has been a subject of study almost since the work of Maxwell.^{39,359} This is because the environment is inhomogeneous, whether it is the atmosphere, the solid earth, or a water system. The inhomogeneity can be uniform in the directionality or it can introduce birefringence by which waves split into different directions depending upon their polarizations. Regardless, wave propagation and their related signal processing methods can give insight into the nature of the media through which the signals are passing, making it a viable tool in many areas of physics and engineering. Here, the Wigner function—along with Ville's work from 1948³⁶⁰—to this day plays an important part.^{39,361–364}

Polarization effects for electromagnetic wave propagation in random media were investigated.³⁶⁵ Based on a mode approach, intermode scattering can be used to introduce the randomness of the medium. The authors studied the loss of polarization that arises from the intermode scattering, with the use of the Wigner function to describe the energy density of the wave. Again, introduction of the Wigner function leads to a set of stochastic equations which describes the propagation and the role of the inhomogeneities. Also using a modal approach, a phase space approximation method for linear dispersive wave propagation with arbitrary initial conditions was developed.³⁶⁶ Numerical solutions of one-dimensional waves were also investigated through discretization of the space.³⁶⁷ Also targeting random media, the acousto-optic effect was studied.³⁶⁸ The radiative transport equations were derived based on the Wigner transform: These describe the propagation of multiply scattered light in a medium whose dielectric permittivity is modulated by an acoustic wave (e.g., the atmosphere-ocean interface). Based on the previous work on a phase-space approach for propagating field-field correlation functions,³⁶⁹ an approach to compute the wave correlation functions inside finite spatial domains driven by complex and statistical sources was introduced.³⁷⁰ An explicit quantitative connection with phase space densities could be made by presenting the correlation function as a Wigner function. Recently, an approach to construct accelerating beams in phase space by designing the corresponding Wigner distribution function was introduced.³⁷¹ Accelerating beams can be realized by light fields that propagate along arbitrary trajectories.³⁷²

It was found that the Wigner function-based method is capable of providing both the initial field distribution and the angular spectrum needed by projecting the Wigner function into the real space and the Fourier space, respectively. The use of Wigner functions to investigate phase and diffraction effects was used for three-dimensional imaging in volumetric scattering media.³⁷³ Here, propagation is described by a simple shearing operation in phase space, allowing one to trace rays (waves) backward through three-dimensional

space to find depth. In Ref. 374, the authors analyzed the phase dynamics of underlying oil-water two-phase flows. Here, the multivariate pseudo Wigner distribution³⁷⁵ was used to calculate the multivariate time-frequency representation for three typical oil-in-water flow patterns. Figure 14 shows the time-frequency representation for a particular flow pattern. Its dominant characteristics lie in slowly quasiperiodic movements and obvious intermittent oscillations.

In other recent work, a time-frequency filter was developed to eliminate cross-terms in the Wigner-Ville distribution, which is an attractive tool—particularly the high resolution it provides—to conduct time-frequency analysis of non-stationary systems.³⁷⁶

Cross-terms manifest when the signal contains nonlinear frequencies or intermittent signal components. A comparison with an instantaneous frequency estimation method based on a continuous wavelet transform validated the proposed method. In related work,³⁷⁷ another approach to reduce cross-terms in the Wigner-Ville distribution using a tunable-Q wavelet transform was introduced. Finally, the Wigner function was used to investigate discrete fractional Fourier transforms using closed-form Hermite–Gaussian-like eigenvectors.³⁷⁸ Here, one should not overlook the close connection between the Wigner functions and the wavelets themselves.³⁹

VIII. CONCLUSION

Formulated over 86 years ago, the Wigner function found broad application in many areas of science and engineering. In a way, it left its most noticeable footprint in quantum information processing and quantum physics, where it facilitates the ability to actually illustrate quantum entanglement: This key feature of the Wigner function is highly important for modern applications, such as in quantum computing and quantum information. But other areas greatly benefited from the unique features as well: From allowing to model quantum transport in electronics, enabling to investigate tunneling in chemistry, and to granting insights

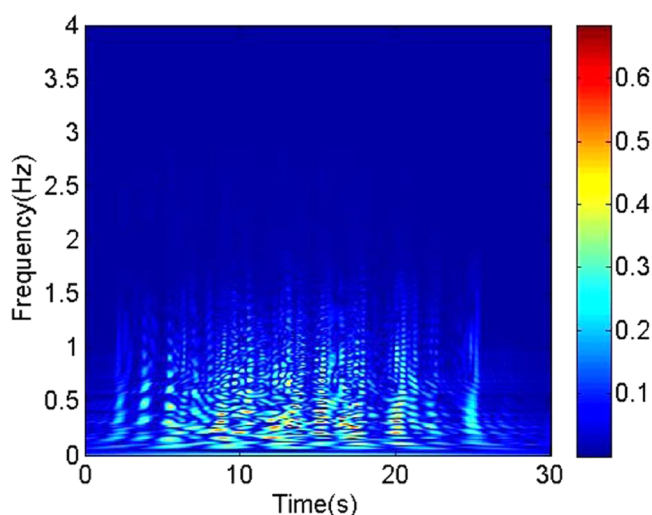


FIG. 14. The multivariate pseudo Wigner distribution of oil-in-water slug flow with the mixture flow velocity 0.0368 m/s and water cut 60%. Reprinted with permission from Gao *et al.*, *Sci. Rep.* 6, 28151 (2016). Copyright 2016 Author(s), licensed under the Creative Commons Attribution 4.0 License.

into matter via signal processing—just to name a few, the Wigner function proved its versatility and importance to many areas of research. The attractiveness of its usage in both the quantum *and* the classical world, as well as the possibility to experimentally reconstruct the Wigner function makes this particular distribution function stand out over other quantum approaches. In particular, it has been demonstrated time and again that the Wigner function can clearly illustrate the exact differences between the classical and the quantum worlds. This inherent attractiveness has been useful to many fields of science over many decades and is still gaining even wider utilization in recent years. Indeed, the vast majority of research dealt with in this review is not older than 2015, underlying this very fact. Moreover, this huge volume of recent work extends over a wide range of fields of modern quantum mechanics, clearly indicating its fruitful applications in modern science.

Moving forward, each area has several challenges and opportunities concerning the application of the Wigner function, a few of these can be stated. In quantum information processing and quantum physics, the challenges in applying the Wigner function are centered around the ability to reconstruct the Wigner function via measurements. Therefore, further improvements to the reconstruction process will enable more detailed understanding (e.g., highlighting entanglement) based on the reconstructed Wigner function. In quantum electronics, self-consistency between solutions of the Wigner equation of motion and solutions of Poisson's equation is of utmost concern as is the computational effort and numerical challenges involved in solving three-dimensional problems and considering scattering effects as well as electromagnetic fields and spins. All of which will be important to conduct research into new device designs operating in the low nanometer regime. In quantum chemistry, calculating the static and dynamical properties of many-body quantum systems is a key challenge. There, the Wigner function has already proved to be a promising option. However, more accurate calculations of, e.g., transport coefficients and rate constants will be necessary and will enable a more accurate prediction of the behavior of many-body systems. In signal processing, among the major challenges is to describe a propagating wave when passing through certain media. The Wigner function is already successfully applied to establish the necessary radiative transport and diffusion equations. Incorporation of additional effects, e.g., optical field polarization, will require further research.

While entanglement was mentioned only in the quantum information area above, every one of these fields depends upon the entanglement. In some areas, new tomographic methods allow the reconstruction of the Wigner function in a manner to demonstrate the entanglement. In other areas, such as quantum electronics, the issue is the destruction of the entanglement by the scattering processes. But, we should add that even in quantum information, the destruction of entanglement sets the time over which quantum computing and information transfer must be done. So, all of these areas can be considered to focus on the presence of entanglement and its coherence time. This fact means that there is a commonality between the areas in which the future focus lies

with problems of improving entanglement and the coherence time.

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- ¹M. Planck, *Ann. Phys.* **309**, 553 (1901).
- ²A. Einstein, *Ann. Phys.* **322**, 132 (1905).
- ³J. J. Thomson, *Proc. Cambridge Philos. Soc.* **14**, 417 (1907).
- ⁴G. I. Taylor, *Proc. Cambridge Philos. Soc.* **15**, 114 (1909).
- ⁵N. Bohr, *London, Edinburgh, Dublin Philos. Mag. J. Sci.* **26**, 1 (1913).
- ⁶L. De Broglie, *Nature* **112**, 540 (1923).
- ⁷W. Heisenberg, *Z. Phys.* **33**, 879 (1925).
- ⁸M. Born and P. Jordan, *Z. Phys.* **34**, 858 (1925).
- ⁹M. Born, W. Heisenberg, and P. Jordan, *Z. Phys.* **35**, 557 (1926).
- ¹⁰E. Schrödinger, *Ann. Phys.* **384**(Ser. 4), 361 (1926).
- ¹¹E. Schrödinger, *Ann. Phys.* **384**(Ser. 4), 489 (1926).
- ¹²E. Schrödinger, *Ann. Phys.* **384**(Ser. 4), 734 (1926).
- ¹³E. Schrödinger, *Ann. Phys.* **385**(Ser. 4), 437 (1926).
- ¹⁴E. Schrödinger, *Ann. Phys.* **386**(Ser. 4), 109 (1926).
- ¹⁵E. Madelung, *Z. Phys.* **40**, 322 (1927).
- ¹⁶E. H. Kennard, *Phys. Rev.* **31**, 876 (1928).
- ¹⁷D. Bohm, *Phys. Rev.* **85**, 166 (1952).
- ¹⁸D. Bohm, *Phys. Rev.* **85**, 180 (1952).
- ¹⁹E. P. Wigner, *Phys. Rev.* **40**, 749 (1932).
- ²⁰H. J. Groenewold, *Physica* **12**, 405 (1946).
- ²¹T. L. Curtright, D. B. Fairlie, and C. K. Zachos, *A Concise Treatise on Quantum Mechanics in Phase Space* (World Scientific, 2014).
- ²²J. E. Moyal, *Math. Proc. Cambridge Philos. Soc.* **45**, 99 (1949).
- ²³G. Baker, *Phys. Rev.* **109**, 2198 (1958).
- ²⁴F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Ann. Phys. (N. Y.)* **111**, 61 (1978).
- ²⁵F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Ann. Phys. (N. Y.)* **111**, 111 (1978).
- ²⁶C. K. Zachos, D. B. Fairlie, and T. L. Curtright, *Quantum Mechanics in Phase Space* (World Scientific, 2005).
- ²⁷D. K. Ferry, *J. Comput. Electron.* **14**, 864 (2015).
- ²⁸E. Colomés, Z. Zhan, and X. Oriols, *J. Comput. Electron.* **14**, 894 (2015).
- ²⁹H.-W. Lee, *Phys. Rep.* **259**, 147 (1995).
- ³⁰A. P. Jauho, C. Flindt, T. Novotný, and A. Donarini, *Phys. Fluids* **17**, 100613 (2005).
- ³¹M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, *Phys. Rep.* **106**, 121 (1984).
- ³²K. Husimi, *Proc. Phys.-Math. Soc. Jpn.* **22**, 264 (1940).
- ³³K. Takahashi, *J. Phys. Soc. Jpn.* **55**, 762 (1986).
- ³⁴L. E. Ballentine, *Quantum Mechanics: A Modern Development* (World Scientific, 1998).
- ³⁵M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, 1990).
- ³⁶N. L. Balazs and B. K. Jennings, *Phys. Rep.* **104**, 347 (1984).
- ³⁷D. Bohm and B. J. Hiley, *The Undivided Universe: An Ontological Interpretation of Quantum Theory* (Routledge, 1994).
- ³⁸G. Albareda, D. Marian, A. Benali, A. Alarcon, S. Moises, and X. Oriols, in *Simul. Transp. Nanodevices*, edited by F. Triozon and P. Dollfus (Wiley-ISTE, 2016), pp. 261–318.
- ³⁹D. K. Ferry and M. Nedjalkov, *The Wigner Function in Science and Technology* (Institute of Physics Publishing, 2018).
- ⁴⁰F. E. Schroeck, *Quantum Mechanics on Phase Space* (Springer, 1996).
- ⁴¹M. Błaszak and Z. Domański, *Ann. Phys. (N. Y.)* **327**, 167 (2012).
- ⁴²W. E. Brittin and W. R. Chappell, *Rev. Mod. Phys.* **34**, 620 (1962).
- ⁴³P. Carruthers and F. Zachariasen, *Rev. Mod. Phys.* **55**, 245 (1983).
- ⁴⁴W. B. Case, *Am. J. Phys.* **76**, 937 (2008).
- ⁴⁵M. A. de Gosson, *The Wigner Transform* (World Scientific, 2017).
- ⁴⁶R. F. O'Connell, *Found. Phys.* **13**, 83 (1983).
- ⁴⁷Y. S. Kim and M. E. Noz, *Phase Space Picture of Quantum Mechanics* (World Scientific, 1991).
- ⁴⁸L. Mandel and E. Wolf, *Rev. Mod. Phys.* **37**, 231 (1965).

- ⁴⁹M. V. Berry, *Philos. Trans. R. Soc., A* **287**, 237 (1977).
- ⁵⁰G. J. Iafrate, H. L. Grubin, and D. K. Ferry, *Phys. Lett. A* **87**, 145 (1982).
- ⁵¹J. R. Barker and S. Murray, *Phys. Lett. A* **93**, 271 (1983).
- ⁵²B. V. K. Vijaya Kumar and C. W. Carroll, *Opt. Eng.* **23**, 236732 (1984).
- ⁵³U. Ravaioli, M. A. Osman, W. Pötz, N. Kluksdahl, and D. K. Ferry, *Phys. B+C* **134**, 36 (1985).
- ⁵⁴W. R. Frensley, *Phys. Rev. B* **36**, 1570 (1987).
- ⁵⁵N. C. Kluksdahl, A. M. Krman, D. K. Ferry, and C. Ringhofer, *Phys. Rev. B* **39**, 7720 (1989).
- ⁵⁶C. Jacoboni and P. Bordone, *Rep. Prog. Phys.* **67**, 1033 (2004).
- ⁵⁷D. Querlioz, P. Dollfus, and M. Mouis, *The Wigner Monte Carlo Method for Nanoelectronic Devices* (John Wiley & Sons, 2013).
- ⁵⁸Y. S. Kim and E. P. Wigner, *Am. J. Phys.* **58**, 439 (1990).
- ⁵⁹Y. Tanimura and S. Mukamel, *J. Phys. Soc. Jpn.* **63**, 66 (1994).
- ⁶⁰K. Na and R. E. Wyatt, *Phys. Scr.* **67**, 169 (2003).
- ⁶¹R. Mazar, *Comput. Struct.* **67**, 119 (1998).
- ⁶²S. Qian and D. Chen, *IEEE Signal Process. Mag.* **16**, 52 (1999).
- ⁶³D. Kohen, C. C. Marston, and D. J. Tannor, *J. Chem. Phys.* **107**, 5236 (1997).
- ⁶⁴T. Curtright, D. Fairlie, and C. Zachos, *Phys. Rev. D* **58**, 025002 (1998).
- ⁶⁵E. W. Piotrowski and J. Śladkowski, *Acta Phys. Pol., B* **32**, 3873 (2001), see <http://www.actaphys.uj.edu.pl/fulltext?series=Reg&vol=32&page=3873>.
- ⁶⁶R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
- ⁶⁷R. P. Rundle, P. W. Mills, T. Tilma, J. H. Samson, and M. J. Everitt, *Phys. Rev. A* **96**, 022117 (2017).
- ⁶⁸S. Deléglise, I. Dotsenko, C. Sayrin, J. Bernu, M. Brune, J. M. Raimond, and S. Haroche, *Nature* **455**, 510 (2008).
- ⁶⁹D. Barral, J. Liñares, and D. Balado, *J. Opt. Soc. Am. B* **33**, 2225 (2016).
- ⁷⁰D. Barral, D. Balado, and J. Liñares, *Photonics* **4**, 2 (2017).
- ⁷¹S. Prabhakar, S. G. Reddy, A. Aadhi, C. Perumangatt, G. K. Samanta, and R. P. Singh, *Phys. Rev. A* **92**, 023822 (2015).
- ⁷²A. Mari and J. Eisert, *Phys. Rev. Lett.* **109**, 230503 (2012).
- ⁷³X. L. Zhao, Z. C. Shi, M. Qin, and X. X. Yi, *Phys. Rev. A* **96**, 013824 (2017).
- ⁷⁴G. Najarbashi and S. Mirzaei, *Opt. Commun.* **377**, 33 (2016).
- ⁷⁵R. McConnell, H. Zhang, J. Hu, S. Cuk, and V. Vuletic, *Nature* **519**, 439 (2015).
- ⁷⁶F.-Z. Siyouri, *Commun. Theor. Phys.* **68**, 729 (2017).
- ⁷⁷J. Joo and E. Ginossar, *Sci. Rep.* **6**, 26338 (2016).
- ⁷⁸M. S. Ateto, *Quantum Inf. Process.* **16**, 267 (2017).
- ⁷⁹M. Jachura, M. Karpiński, K. Banaszek, D. Bharadwaj, J. Lugani, and K. Thyagarajan, *Phys. Rev. A* **95**, 032322 (2017).
- ⁸⁰W. Pfaff, C. J. Axline, L. D. Burkhardt, U. Vool, P. Reinhold, L. Frunzio, L. Jiang, M. H. Devoret, and R. J. Schoelkopf, *Nat. Phys.* **13**, 882 (2017).
- ⁸¹F. Xu, C. C. Martens, and Y. Zheng, *Phys. Rev. A* **96**, 022138 (2017).
- ⁸²M. Nedjalkov, J. Weinbub, and D. K. Ferry, *J. Comput. Electron.* **14**, 857 (2015).
- ⁸³A. Royer, *Phys. Rev. A* **43**, 44 (1991).
- ⁸⁴A. Jannussis, A. Streklas, and K. Vlachos, *Phys. A* **107**, 587 (1981).
- ⁸⁵A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**, 777 (1935).
- ⁸⁶E. Schrödinger, *Math. Proc. Cambridge Philos. Soc.* **31**, 555 (1935).
- ⁸⁷A. M. O. de Almeida, in *Entanglement and Decoherence*, edited by A. Buchleitner, C. Viviescas, and M. Tiersch (Springer, 2009), pp. 157–219.
- ⁸⁸A. Ourjoumte, R. Tualle-Brouiri, J. Laurat, and P. Grangier, *Science* **312**, 83 (2006).
- ⁸⁹D. Bohm, *Quantum Theory* (Prentice-Hall, 1951).
- ⁹⁰W. P. Schleich, *Quantum Optics in Phase Space* (WILEY, 2001).
- ⁹¹S. Olivares, *Eur. Phys. J.: Spec. Top.* **203**, 3 (2012).
- ⁹²M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, 1997).
- ⁹³D. Dragoman, *Prog. Opt.* **37**, 1 (1997).
- ⁹⁴M. Testorf, B. Hennelly, and J. Ojeda-Castaneda, *Phase-Space Optics* (McGraw-Hill Education, 2009).
- ⁹⁵J. C. Petrucci and M. A. Alonso, “The Wigner function in optics,” in *The Optics Encyclopedia* (Wiley-VCH, 2015), p. 1.
- ⁹⁶M. J. Bastiaans, *J. Opt. Soc. Am.* **69**, 1710 (1979).
- ⁹⁷A. Walther, *J. Opt. Soc. Am.* **58**, 1256 (1968).
- ⁹⁸G. H. Yuan, E. T. Rogers, and N. I. Zheludev, *Light Sci. Appl.* **6**, e17036 (2017).
- ⁹⁹E. Maguid, I. Yulevich, M. Yannai, V. Kleiner, M. L. Brongersma, and E. Hasman, *Light Sci. Appl.* **6**, e17027 (2017).
- ¹⁰⁰E. T. Jaynes and F. W. Cummings, *Proc. IEEE* **51**, 89 (1963).
- ¹⁰¹R. Birrittella, K. Cheng, and C. C. Gerry, *Opt. Commun.* **354**, 286 (2015).
- ¹⁰²O. de los Santos-Sánchez, J. Récamier, and R. Jáuregui, *Phys. Scr.* **90**, 074018 (2017).
- ¹⁰³M. Ghorbani, M. J. Faghihi, and H. Safari, *J. Opt. Soc. Am. B* **34**, 1884 (2017).
- ¹⁰⁴M. Pandit, S. Das, S. S. Roy, H. S. Dhar, and U. Sen, *J. Phys. B: At., Mol. Opt. Phys.* **51**, 045501 (2018).
- ¹⁰⁵E. Faraji and M. K. Tavassoly, *Opt. Commun.* **354**, 333 (2015).
- ¹⁰⁶D. Lv, S. An, M. Um, J. Zhang, J. N. Zhang, M. S. Kim, and K. Kim, *Phys. Rev. A* **95**, 043813 (2017).
- ¹⁰⁷L. T. Shen, Z. B. Yang, H. Z. Wu, and S. B. Zheng, *Phys. Rev. A* **93**, 063837 (2016).
- ¹⁰⁸A. Miranowicz, J. Bajer, N. Lambert, Y. X. Liu, and F. Nori, *Phys. Rev. A* **93**, 013808 (2016).
- ¹⁰⁹X. Gu, A. F. Kockum, A. Miranowicz, Y. Xi Liu, and F. Nori, *Phys. Rep.* **718–719**, 1 (2017).
- ¹¹⁰Y. Perl, Y. B. Band, and E. Boukobza, *Phys. Rev. A* **95**, 053823 (2017).
- ¹¹¹S. Dambach, B. Kubala, and J. Ankerhold, *Fortschr. Phys.* **65**, 1600061 (2017).
- ¹¹²A. D. Armour, B. Kubala, and J. Ankerhold, *Phys. Rev. B* **91**, 184508 (2015).
- ¹¹³R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, *Phys. Rev. Lett.* **55**, 2409 (1985).
- ¹¹⁴L. A. Wu, H. J. Kimble, J. L. Hall, and H. Wu, *Phys. Rev. Lett.* **57**, 2520 (1986).
- ¹¹⁵X. X. Xu and H. C. Yuan, *Quantum Inf. Process.* **14**, 411 (2015).
- ¹¹⁶F. Albarelli, A. Ferraro, M. Paternostro, and M. G. A. Paris, *Phys. Rev. A* **93**, 032112 (2016).
- ¹¹⁷G. Ren, J. M. Du, H. J. Yu, and W. H. Zhang, *Optik* **127**, 3828 (2016).
- ¹¹⁸T. Yang, X. Ji, X. Li, H. Zhang, and T. Wang, *Opt. Commun.* **359**, 146 (2016).
- ¹¹⁹S. Xu, X. X. Xu, C. J. Liu, H. L. Zhang, and L. Y. Hu, *Optik* **144**, 664 (2017).
- ¹²⁰R. Ramírez and M. Reboiro, *Phys. Lett. A* **380**, 1117 (2016).
- ¹²¹S. Shakeri, Z. Mahmoudi, M. H. Zandi, and A. R. Bahrampour, *Opt. Commun.* **370**, 55 (2016).
- ¹²²C. S. Muñoz, A. Lara, J. Puebla, and F. Nori, *Phys. Rev. Lett.* **121**, 123604 (2018).
- ¹²³C. H. Lv, Y. X. Cai, and Y. W. Wang, *Optik* **127**, 4057 (2016).
- ¹²⁴H. Le Jeannic, A. Cavaillès, K. Huang, R. Filip, and J. Laurat, *Phys. Rev. Lett.* **120**, 073603 (2018).
- ¹²⁵H. C. F. Lemos, A. C. L. Almeida, B. Amaral, and A. C. Oliveira, *Phys. Lett. A* **382**, 823 (2018).
- ¹²⁶X. Deng, S. Hao, H. Guo, C. Xie, and X. Su, *Sci. Rep.* **6**, 22914 (2016).
- ¹²⁷S. Rahimi-Keshari, T. C. Ralph, and C. M. Caves, *Phys. Rev. X* **6**, 021039 (2016).
- ¹²⁸H. Liang Zhang, H. Chun Yuan, L. Yun Hu, and X. Xiang Xu, *Opt. Commun.* **356**, 223 (2015).
- ¹²⁹K. Marshall, C. S. Jacobsen, C. Schäfermeier, T. Gehring, C. Weedbrook, and U. L. Andersen, *Nat. Commun.* **7**, 13795 (2016).
- ¹³⁰R. Schnabel, *Phys. Rep.* **684**, 1 (2017).
- ¹³¹M. A. Taylor and W. P. Bowen, *Phys. Rep.* **615**, 1 (2016).
- ¹³²P. D. Drummond, *Phys. Rev. A* **96**, 062104 (2017).
- ¹³³D. Bouwmeester, A. K. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer, 2000).
- ¹³⁴G. S. Agarwal, R. R. Puri, and R. P. Singh, *Phys. Rev. A* **56**, 4207 (1997).
- ¹³⁵Y. Z. Li, F. Jia, H. L. Zhang, J. H. Huang, and L. Y. Hu, *Laser Phys. Lett.* **12**, 115203 (2015).
- ¹³⁶M. Walschaers, C. Fabre, V. Parigi, and N. Treps, *Phys. Rev. Lett.* **119**, 183601 (2017).
- ¹³⁷M. Walschaers, C. Fabre, V. Parigi, and N. Treps, *Phys. Rev. A* **96**, 053835 (2017).
- ¹³⁸C. Baune, J. Fiurásek, and R. Schnabel, *Phys. Rev. A* **95**, 061802 (2017).
- ¹³⁹K. Thapliyal, N. L. Samantray, J. Banerji, and A. Pathak, *Phys. Lett. A* **381**, 3178 (2017).
- ¹⁴⁰D.-M. Lu, *Int. J. Theor. Phys.* **56**, 3514 (2017).
- ¹⁴¹S. Puri, C. K. Andersen, A. L. Grimsom, and A. Blais, *Nat. Commun.* **8**, 15785 (2017).
- ¹⁴²G. A. Brawley, M. R. Vanner, P. E. Larsen, S. Schmid, A. Boisen, and W. P. Bowen, *Nat. Commun.* **7**, 10988 (2016).
- ¹⁴³M. Ringbauer, T. J. Weinhold, L. A. Howard, A. G. White, and M. R. Vanner, *New J. Phys.* **20**, 053042 (2018).
- ¹⁴⁴K. E. Khosla, M. R. Vanner, N. Ares, and E. A. Laird, *Phys. Rev. X* **8**, 021052 (2017).

- ¹⁴⁵M. A. Lemonde, N. Didier, and A. A. Clerk, *Nat. Commun.* **7**, 11338 (2016).
- ¹⁴⁶A. A. Rakhubovsky and R. Filip, *Sci. Rep.* **7**, 46764 (2017).
- ¹⁴⁷M. Das, J. K. Verma, and P. K. Pathak, *Phys. Rev. A* **96**, 033837 (2017).
- ¹⁴⁸T. Weiss, S. Walter, and F. Marquardt, *Phys. Rev. A* **95**, 041802(R) (2017).
- ¹⁴⁹K. Ishibashi and R. Kanamoto, *Phys. Rev. E* **96**, 052210 (2017).
- ¹⁵⁰R. W. Andrews, A. P. Reed, K. Cicak, J. D. Teufel, and K. W. Lehnert, *Nat. Commun.* **6**, 10021 (2015).
- ¹⁵¹G. Harder, C. Silberhorn, J. Rehacek, Z. Hradil, L. Motka, B. Stoklasa, and L. L. Sánchez-Soto, *Phys. Rev. Lett.* **116**, 133601 (2016).
- ¹⁵²H. M. Li, X. X. Xu, H. C. Yuan, and Z. Wang, *Chin. Phys. B* **25**, 104203 (2016).
- ¹⁵³C. Porto, D. Rusca, S. Cialdi, A. Crespi, R. Osellame, D. Tamascelli, S. Olivares, and M. G. A. Paris, *J. Opt. Soc. Am. B* **35**, 1596 (2018).
- ¹⁵⁴E. Knyazev, K. Y. Spasibko, M. V. Chekhova, and F. Y. Khalili, *New J. Phys.* **20**, 013005 (2018).
- ¹⁵⁵K. Y. Spasibko, M. V. Chekhova, and F. Y. Khalili, *Phys. Rev. A* **96**, 023822 (2017).
- ¹⁵⁶D. Ferraro, B. Roussel, C. Cabart, E. Thibierge, G. Fève, C. Grenier, and P. Degiovanni, *Phys. Rev. Lett.* **113**, 166403 (2014).
- ¹⁵⁷D. Ferraro, T. Jonckheere, J. Rech, and T. Martin, *Phys. Status Solidi B* **254**, 1600531 (2017).
- ¹⁵⁸A. Marguerite, E. Bocquillon, J. M. Berroir, B. Plaçais, A. Cavanna, Y. Jin, P. Degiovanni, and G. Fève, *Phys. Status Solidi B* **254**, 1600618 (2017).
- ¹⁵⁹C. Bäuerle, D. C. Glattli, T. Meunier, F. Portier, P. Roche, P. Rouleau, S. Takada, and X. Waintal, *Rep. Prog. Phys.* **81**, 056503 (2018).
- ¹⁶⁰X. X. Xu, H. C. Yuan, and S. J. Ma, *J. Opt. Soc. Am. B* **33**, 1322 (2016).
- ¹⁶¹I. Straka, L. Lachman, J. Hloušek, M. Miková, M. Mičuda, M. Ježek, and R. Filip, *npj Quantum Inf.* **4**, 4 (2018).
- ¹⁶²J. R. Souquet and A. A. Clerk, *Phys. Rev. A* **93**, 060301 (2016).
- ¹⁶³P. Zapletal and R. Filip, *Sci. Rep.* **7**, 1484 (2017).
- ¹⁶⁴O. Di Matteo, L. L. Sánchez-Soto, G. Leuchs, and M. Grassl, *Phys. Rev. A* **95**, 022340 (2017).
- ¹⁶⁵O. Landon-Cardinal, L. C. G. Govia, and A. A. Clerk, *Phys. Rev. Lett.* **120**, 090501 (2018).
- ¹⁶⁶C. Kupchak, S. Rind, B. Jordaán, and E. Figueroa, *Sci. Rep.* **5**, 16581 (2015).
- ¹⁶⁷M. Mirhosseini, O. S. Magaña-Loaiza, C. Chen, S. M. Hashemi Rafsanjani, and R. W. Boyd, *Phys. Rev. Lett.* **116**, 130402 (2016).
- ¹⁶⁸C. R. Müller, C. Peuntinger, T. Dirmeier, I. Khan, U. Vogl, C. Marquardt, G. Leuchs, L. L. Sánchez-Soto, Y. S. Teo, Z. Hradil, and J. Řeháček, *Phys. Rev. Lett.* **117**, 070801 (2016).
- ¹⁶⁹S. Cialdi, C. Porto, D. Cipriani, S. Olivares, and M. G. A. Paris, *Phys. Rev. A* **93**, 043805 (2016).
- ¹⁷⁰X. Chen, C. Ding, H. Pan, K. Huang, J. Laurat, G. Wu, and E. Wu, *Sci. Rep.* **7**, 44600 (2017).
- ¹⁷¹J. Y. Haw, J. Zhao, J. Dias, S. M. Assad, M. Bradshaw, R. Blandino, T. Symul, T. C. Ralph, and P. K. Lam, *Nat. Commun.* **7**, 13222 (2016).
- ¹⁷²Q. Hou, W. Yang, C. Chen, and Z. Yin, *Sci. Rep.* **6**, 37542 (2016).
- ¹⁷³H. Goto, *Sci. Rep.* **6**, 21686 (2016).
- ¹⁷⁴F. Minganti, N. Bartolo, J. Lolli, W. Casteels, and C. Ciuti, *Sci. Rep.* **6**, 26987 (2016).
- ¹⁷⁵J. K. Eastman, J. J. Hope, and A. R. R. Carvalho, *Sci. Rep.* **7**, 44684 (2017).
- ¹⁷⁶T. Zhang, J. Wei, and Q. Wang, *Commun. Theor. Phys.* **67**, 391 (2017).
- ¹⁷⁷D. V. Sychev, A. E. Ulanov, A. A. Pushkina, M. W. Richards, I. A. Fedorov, and A. I. Lvovsky, *Nat. Photonics* **11**, 379 (2017).
- ¹⁷⁸D. V. Karlovets and V. G. Serbo, *Phys. Rev. Lett.* **119**, 173601 (2017).
- ¹⁷⁹R. Chakrabarti and V. Yogesh, *J. Phys. B: At., Mol. Opt. Phys.* **49**, 075502 (2016).
- ¹⁸⁰M. Balamurugan, R. Chakrabarti, and B. V. Jenisha, *Phys. A* **473**, 428 (2017).
- ¹⁸¹R. Chakrabarti and V. Yogesh, *Phys. A* **490**, 886 (2018).
- ¹⁸²C. Joshi, E. K. Irish, and T. P. Spiller, *Sci. Rep.* **7**, 45587 (2017).
- ¹⁸³C. R. Müller, G. Leuchs, C. Marquardt, and U. L. Andersen, *Phys. Rev. A* **96**, 042311 (2017).
- ¹⁸⁴M. Howard, J. Wallman, V. Veitch, and J. Emerson, *Nature* **510**, 351 (2014).
- ¹⁸⁵N. Delfosse, P. Allard Guerin, J. Bian, and R. Raussendorf, *Phys. Rev. X* **5**, 021003 (2015).
- ¹⁸⁶R. Raussendorf, D. E. Browne, N. Delfosse, C. Okay, and J. Bermejo-Vega, *Phys. Rev. A* **95**, 052334 (2017).
- ¹⁸⁷J. S. Bell, *Phys. Phys. Fiz.* **1**, 195 (1964).
- ¹⁸⁸G. Boole, *An Investigation of the Laws of Thought* (Walton and Maberly, 1854).
- ¹⁸⁹K. P. Seshadreesan, J. P. Dowling, and G. S. Agarwal, *Phys. Scr.* **90**, 074029 (2015).
- ¹⁹⁰K. Makino, Y. Hashimoto, J. I. Yoshikawa, H. Ohdan, T. Toyama, P. van Loock, and A. Furusawa, *Sci. Adv.* **2**, e1501772 (2016).
- ¹⁹¹Y. Xiang, B. Xu, L. Mišta, T. Tufarelli, Q. He, and G. Adesso, *Phys. Rev. A* **96**, 042326 (2017).
- ¹⁹²H. F. Dowker and J. J. Halliwell, *Phys. Rev. D* **46**, 1580 (1992).
- ¹⁹³B. Sokolov, I. Vilja, and S. Maniscalco, *Phys. Rev. A* **96**, 012126 (2017).
- ¹⁹⁴C. Weiss, S. L. Cornish, S. A. Gardiner, and H. P. Breuer, *Phys. Rev. A* **93**, 013605 (2016).
- ¹⁹⁵D. K. Ferry, R. Akis, and R. Brunner, *Phys. Scr.* **T165**, 014010 (2015).
- ¹⁹⁶T. Tilma, M. J. Everitt, J. H. Samson, W. J. Munro, and K. Nemoto, *Phys. Rev. Lett.* **117**, 180401 (2016).
- ¹⁹⁷R. Cabrera, D. I. Bondar, K. Jacobs, and H. A. Rabitz, *Phys. Rev. A* **92**, 042122 (2015).
- ¹⁹⁸H. Yang, R. N. Rutte, L. Jones, M. Simson, R. Sagawa, H. Ryll, M. Huth, T. J. Pannycok, M. L. H. Green, H. Soltan, Y. Kondo, B. G. Davis, and P. D. Nellist, *Nat. Commun.* **7**, 12532 (2016).
- ¹⁹⁹A. Rudenko, L. Inhester, K. Hanasaki, X. Li, S. J. Robatjazi, B. Erk, R. Boll, K. Toyota, Y. Hao, O. Vendrell, C. Bomme, E. Savelyev, B. Rudek, L. Foucar, S. H. Southworth, C. S. Lehmann, B. Kraessig, T. Marchenko, M. Simon, K. Ueda, K. R. Ferguson, M. Bucher, T. Gorkhover, S. Carron, R. Alonso-Mori, J. E. Koglin, J. Correa, G. J. Williams, S. Boutet, L. Young, C. Bostedt, S. K. Son, R. Santra, and D. Rolles, *Nature* **546**, 129 (2017).
- ²⁰⁰D. Leiner, R. Zeier, and S. J. Glaser, *Phys. Rev. A* **96**, 063413 (2017).
- ²⁰¹M. Kolodrubetz, B. M. Fregoso, and J. E. Moore, *Phys. Rev. B* **94**, 195124 (2016).
- ²⁰²Y. Tian, Z. Wang, P. Zhang, G. Li, J. Li, and T. Zhang, *Phys. Rev. A* **97**, 013840 (2018).
- ²⁰³D. Bossini, S. Dal Conte, Y. Hashimoto, A. Secchi, R. V. Pisarev, T. Rasing, G. Cerullo, and A. V. Kimel, *Nat. Commun.* **7**, 10645 (2016).
- ²⁰⁴O. Jonasson and I. Knezevic, *J. Comput. Electron.* **14**, 879 (2015).
- ²⁰⁵S. Mei, Y. Shi, O. Jonasson, and I. Knezevic, in *Handb. Optoelectron. Device Model. Simulation* (CRC Press, 2017), Vol. 2, Part VI.
- ²⁰⁶S. Kohler, T. Dittrich, and P. Hänggi, *Phys. Rev. E* **55**, 300 (1997).
- ²⁰⁷D. Campos, *Pramana* **88**, 54 (2017).
- ²⁰⁸J. F. G. Santos and A. E. Bernardini, *Phys. A* **445**, 75 (2016).
- ²⁰⁹M. Prado Reynoso, P. C. López Vázquez, and T. Gorin, *Phys. Rev. A* **95**, 022118 (2017).
- ²¹⁰P. A. Rose, A. C. McClung, T. E. Keating, A. T. C. Steege, E. S. Egge, and A. K. Pattanayak, *Pramana* **87**, 32 (2016).
- ²¹¹L. Orr, L. Hernández de la Peña, and P.-N. N. Roy, *J. Chem. Phys.* **146**, 214116 (2017).
- ²¹²D. Kienzler, C. Flühmann, V. Negnevitsky, H. Y. Lo, M. Marinelli, D. Nadlinger, and J. P. Home, *Phys. Rev. Lett.* **116**, 140402 (2016).
- ²¹³S. Ding, G. Maslennikov, R. Häblützel, H. Loh, and D. Matsukevich, *Phys. Rev. Lett.* **119**, 150404 (2017).
- ²¹⁴M. Um, J. Zhang, D. Lv, Y. Lu, S. An, J. N. Zhang, H. Nha, M. S. Kim, and K. Kim, *Nat. Commun.* **7**, 11410 (2016).
- ²¹⁵J. F. Corney and M. K. Olsen, *Phys. Rev. A* **91**, 023824 (2015).
- ²¹⁶P. D. Drummond and B. Opanchuk, *Phys. Rev. A* **96**, 043616 (2017).
- ²¹⁷A. Ivanov and H. P. Breuer, *Phys. Rev. E* **95**, 042115 (2017).
- ²¹⁸O. L. Acevedo, A. Safavi-Naini, J. Schachenmayer, M. L. Wall, R. Nandkishore, and A. M. Rey, *Phys. Rev. A* **96**, 033604 (2017).
- ²¹⁹A. Piñeiro Orioli, A. Safavi-Naini, M. L. Wall, and A. M. Rey, *Phys. Rev. A* **96**, 033607 (2017).
- ²²⁰R. Avriell, B. Murr, and F. Pistolesi, *Phys. Rev. B* **97**, 155414 (2018).
- ²²¹E. B. Balutsev, I. V. Molodtsova, and P. Schuck, *Phys. At. Nucl.* **80**, 86 (2017).
- ²²²F. Ronetti, L. Vannucci, D. Ferraro, T. Jonckheere, J. Rech, T. Martin, and M. Sassetti, *Phys. Rev. B* **98**, 075401 (2018).
- ²²³D. K. Ferry, *Transport in Semiconductor Mesoscopic Devices* (IOP Publishing, 2015).
- ²²⁴Y. X. Liu, L. F. Wei, and F. Nori, *Phys. Rev. B* **72**, 014547 (2005).
- ²²⁵A. B. A. Mohamed, *Phys. Lett. A* **374**, 4115 (2010).
- ²²⁶E. T. Holland, B. Vlastakis, R. W. Heeres, M. J. Reagor, U. Vool, Z. Leghtas, L. Frunzio, G. Kirchmair, M. H. Devoret, M. Mirrahimi, and R. J. Schoelkopf, *Phys. Rev. Lett.* **115**, 180501 (2015).
- ²²⁷M. Reboiro, O. Civitarese, R. Ramirez, and D. Tielas, *Phys. Scr.* **92**, 094004 (2017).

- ²²⁸B. Yang and S. Yao, *Phys. Rev. A* **43**, 1983 (1991).
- ²²⁹A. S. Larkin, V. S. Filinov, and V. E. Fortov, *Contrib. Plasma Phys.* **56**, 187 (2016).
- ²³⁰V. S. Filinov, Y. B. Ivanov, V. E. Fortov, M. Bonitz, and P. R. Levashov, *Phys. Rev. C* **87**, 035207 (2013).
- ²³¹G. Brodin, R. Ekman, and J. Zamanian, *Plasma Phys. Controlled Fusion* **59**, 014043 (2017).
- ²³²A. S. Larkin and V. S. Filinov, *Contrib. Plasma Phys.* **58**, 107 (2018).
- ²³³J. Rosato, *Ann. Phys. (N. Y.)* **383**, 130 (2017).
- ²³⁴P. Massignan, A. Lampo, J. Wehr, and M. Lewenstein, *Phys. Rev. A* **91**, 033627 (2015).
- ²³⁵J. P. Mallarino and G. Téllez, *Phys. Rev. E* **91**, 062140 (2015).
- ²³⁶A. S. Larkin, V. S. Filinov, and V. E. Fortov, *J. Phys. A: Math. Theor.* **51**, 035002 (2018).
- ²³⁷J. Rosato, *Eur. Phys. J. D* **71**, 28 (2017).
- ²³⁸Y. V. Slyusarenko and O. Y. Slyusarenko, *J. Math. Phys.* **58**, 113302 (2017).
- ²³⁹O. Klein, *Z. Phys.* **37**, 895 (1926).
- ²⁴⁰W. Gordon, *Z. Phys.* **40**, 117 (1926).
- ²⁴¹V. Fock, *Z. Phys.* **39**, 226 (1926).
- ²⁴²V. Fock, *Z. Phys.* **38**, 242 (1926).
- ²⁴³P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, 1930).
- ²⁴⁴D. K. Ferry, *An Introduction to Quantum Transport in Semiconductors* (Pan Stanford, 2017).
- ²⁴⁵X. Li Sheng, D. H. Rischke, D. Vasak, and Q. Wang, *Eur. Phys. J. A* **54**, 21 (2018).
- ²⁴⁶R. Ekman, F. A. Asenjo, and J. Zamanian, *Phys. Rev. E* **96**, 023207 (2017).
- ²⁴⁷M. Rashki and S. Jalalzadeh, *Gen. Relativ. Gravitation* **49**, 14 (2017).
- ²⁴⁸R. H. Fang, J. Y. Pang, Q. Wang, and X. N. Wang, *Phys. Rev. D* **95**, 014032 (2017).
- ²⁴⁹J. H. Gao, S. Pu, and Q. Wang, *Phys. Rev. D* **96**, 016002 (2017).
- ²⁵⁰M. N. Chernodub and M. A. Zubkov, *Phys. Rev. D* **96**, 056006 (2017).
- ²⁵¹M. Engelhardt, *Phys. Rev. D* **95**, 094505 (2017).
- ²⁵²C.-Q. Guo, C.-J. Zhang, and J. Xu, *Eur. Phys. J. A* **53**, 233 (2017).
- ²⁵³H. Tsukiji, H. Iida, T. Kunihiro, A. Ohnishi, and T. T. Takahashi, *Phys. Rev. D* **94**, 091502 (2016).
- ²⁵⁴T. Koide and T. Kodama, *J. Phys. G: Nucl. Part. Phys.* **43**, 095103 (2016).
- ²⁵⁵Y. Hidaka, S. Pu, and D.-L. Yang, *Phys. Rev. D* **95**, 091901 (2017).
- ²⁵⁶S. Jeon and T. Epelbaum, *Ann. Phys. (N. Y.)* **364**, 1 (2016).
- ²⁵⁷R. J. Fries, K. Han, and C. M. Ko, *Nucl. Phys. A* **956**, 601 (2016).
- ²⁵⁸T. Song, J. Aichelin, and E. Bratkovskaya, *Phys. Rev. C* **96**, 014907 (2017).
- ²⁵⁹B. Chen and J. Zhao, *Phys. Lett. B* **772**, 819 (2017).
- ²⁶⁰J. More, A. Mukherjee, and S. Nair, *Phys. Rev. D* **95**, 074039 (2017).
- ²⁶¹D. Chakrabarti, T. Maji, C. Mondal, and A. Mukherjee, *Phys. Rev. D* **95**, 074028 (2017).
- ²⁶²Y. Hidaka, S. Pu, and D. L. Yang, *Phys. Rev. D* **97**, 016004 (2018).
- ²⁶³G. Manfredi, O. Morandi, L. Friedland, T. Jenke, and H. Abele, *Phys. Rev. D* **95**, 025016 (2017).
- ²⁶⁴J. T. Mendonça and L. Stenflo, *Phys. Scr.* **90**, 055001 (2015).
- ²⁶⁵S. Das, M. P. G. Robbins, and M. A. Walton, *Can. J. Phys.* **94**, 139 (2016).
- ²⁶⁶F. Benatti, R. Floreanini, S. Olivares, and E. Sindici, *Int. J. Quantum Inf.* **15**, 1740014 (2017).
- ²⁶⁷O. Jonasson, "Quantum transport in semiconductor heterostructures using density-matrix and Wigner-function formalisms," Ph.D. thesis (University of Wisconsin-Madison, WI, USA, 2016), see https://homepages.cae.wisc.edu/~knezevic/pdfs/Olafur_Jonasson_Dissertation_2016.pdf.
- ²⁶⁸P. Ellinghaus, "Two-dimensional Wigner Monte Carlo simulation for time-resolved quantum transport with scattering," Ph.D. thesis (TU Wien, Austria, 2016), see <http://www.ue.tuwien.ac.at/phd/ellinghaus/>.
- ²⁶⁹C. Jacoboni, *Theory of Electron Transport in Semiconductors* (Springer, 2010).
- ²⁷⁰M. Nedjalkov, D. Querlioz, P. Dollfus, and H. Kosina, in *Nano-Electronic Devices*, edited by D. Vasilevski and S. M. Goodnick (Springer, 2011), pp. 289–358, see http://dx.doi.org/10.1007/978-1-4419-8840-9_5.
- ²⁷¹P. Dollfus, D. Querlioz, and J. Saint Martin, in *Simul. Transp. Nanodevices*, edited by F. Triozon and P. Dollfus (Wiley-ISTE, 2016), pp. 319–369, see <http://dx.doi.org/10.1002/9781118761793.ch8>.
- ²⁷²C. Ringhofer, D. K. Ferry, and N. Klusdahl, *Transp. Theory Stat. Phys.* **18**, 331 (1989).
- ²⁷³K. Y. Kim, J. Kim, and S. Kim, *AIP Adv.* **6**, 065314 (2016).
- ²⁷⁴K. Y. Kim, S. Kim, and T. Tang, *J. Comput. Electron.* **16**, 148 (2017).
- ²⁷⁵K. Y. Kim and S. Kim, *Solid-State Electron.* **111**, 22 (2015).
- ²⁷⁶A. Dorda and F. Schürer, *J. Comput. Phys.* **284**, 95 (2015).
- ²⁷⁷S. Datta, *Electronic Transport in Mesoscopic Systems* (Cambridge University Press, 1997).
- ²⁷⁸C. Ringhofer, *SIAM J. Numer. Anal.* **27**, 32 (1990).
- ²⁷⁹O. Furtmaier, S. Succi, and M. Mendoza, *J. Comput. Phys.* **305**, 1015 (2016).
- ²⁸⁰Y. Xiong, Z. Chen, and S. Shao, *SIAM J. Sci. Comput.* **38**, B491 (2016).
- ²⁸¹A. Thomann and A. Borzi, *Numer. Methods Partial Differ. Equations* **33**, 62 (2017).
- ²⁸²M. L. Van de Put, B. Sorée, and W. Magnus, *J. Comput. Phys.* **350**, 314–325 (2017).
- ²⁸³L. Schulz and D. Schulz, *IEEE Trans. Nanotechnol.* **15**, 801 (2016).
- ²⁸⁴J.-H. Lee and M. Shin, *IEEE Trans. Nanotechnol.* **16**, 1028 (2017).
- ²⁸⁵G. Bastard, J. A. Brum, and R. Ferreira, in *Semicond. Heterostruct. Nanostructures*, edited by H. Ehrenreich and D. Turnbull (Academic Press, 1991), pp. 229–415, see [http://dx.doi.org/10.1016/S0081-1947\(08\)60092-2](http://dx.doi.org/10.1016/S0081-1947(08)60092-2).
- ²⁸⁶N. Klusdahl, W. Pötz, U. Ravaioli, and D. K. Ferry, *Superlattices Microstruct.* **3**, 41 (1987).
- ²⁸⁷I. Dimov, M. Nedjalkov, J. M. Sellier, and S. Selberherr, *J. Comput. Electron.* **14**, 859 (2015).
- ²⁸⁸D. Schulz and A. Mahmood, *IEEE J. Quantum Electron.* **52**, 8700109 (2016).
- ²⁸⁹F. Rossi, P. Poli, and C. Jacoboni, *Semicond. Sci. Technol.* **7**, 1017 (1992).
- ²⁹⁰K. Hess, *Monte Carlo Device Simulation* (Springer, 1991).
- ²⁹¹O. Muscato and W. Wagner, *SIAM J. Sci. Comput.* **38**, A1483 (2016).
- ²⁹²C. Jacoboni, *The Monte Carlo Method for Semiconductor Device Simulation* (Springer, 1989).
- ²⁹³K. Tomizawa, *Numerical Simulation of Submicron Semiconductor Devices* (Artech House Publishers, 1993).
- ²⁹⁴J. M. Sellier, M. Nedjalkov, and I. Dimov, *Phys. Rep.* **577**, 1 (2015).
- ²⁹⁵L. Shifren, C. Ringhofer, and D. K. Ferry, *IEEE Trans. Electron Devices* **50**, 769 (2003).
- ²⁹⁶P. Ellinghaus, J. Weinbub, M. Nedjalkov, S. Selberherr, and I. Dimov, *J. Comput. Electron.* **14**, 151 (2015).
- ²⁹⁷J. Weinbub, P. Ellinghaus, and M. Nedjalkov, *J. Comput. Electron.* **14**, 922 (2015).
- ²⁹⁸S. Solórzano, M. Mendoza, S. Succi, and H. J. Herrmann, *Phys. Rev. E* **97**, 013308 (2018).
- ²⁹⁹M. Nedjalkov, S. Selberherr, D. K. Ferry, D. Vasilevski, P. Dollfus, D. Querlioz, I. Dimov, and P. Schwaha, *Ann. Phys. (N. Y.)* **328**, 220 (2013).
- ³⁰⁰M. Nedjalkov, P. Ellinghaus, J. Weinbub, T. Sadi, A. Asenov, I. Dimov, and S. Selberherr, *Comput. Phys. Commun.* **228**, 30 (2018).
- ³⁰¹R. C. Iotti, F. Dolcini, and F. Rossi, *Phys. Rev. B* **96**, 115420 (2017).
- ³⁰²D. K. Ferry and I. Welland, *J. Comput. Electron.* **17**, 110 (2018).
- ³⁰³Z. Zhan, E. Colomés, and X. Oriols, *J. Comput. Electron.* **15**, 1206 (2016).
- ³⁰⁴E. Colomés, Z. Zhan, D. Marian, and X. Oriols, *Phys. Rev. B* **96**, 075135 (2017).
- ³⁰⁵B. J. Spisak, M. Wołoszyn, and D. Szydłowski, *J. Comput. Electron.* **14**, 916 (2015).
- ³⁰⁶A. Kommini and Z. Aksamija, *J. Phys.: Condens. Matter* **30**, 044004 (2018).
- ³⁰⁷G. J. Iafrate, V. N. Sokolov, and J. B. Krieger, *Phys. Rev. B* **96**, 144303 (2017).
- ³⁰⁸M. Nedjalkov, J. Weinbub, P. Ellinghaus, and S. Selberherr, *J. Comput. Electron.* **14**, 888 (2015).
- ³⁰⁹M. Nedjalkov, J. Weinbub, M. Ballicchia, S. Selberherr, I. Dimov, and D. K. Ferry, "Wigner equation for general electromagnetic fields: The Weyl-Stratonovich transform" (submitted).
- ³¹⁰M. Wołoszyn and B. J. Spisak, *Phys. Rev. B* **96**, 075440 (2017).
- ³¹¹O. Jonasson and I. Knezevic, *Phys. Rev. B* **90**, 165415 (2014).
- ³¹²J.-H. Lee, M. Shin, S.-J. Byun, and W. Kim, *J. Korean Phys. Soc.* **72**, 622 (2018).
- ³¹³H. Jiang, W. Cai, and R. Tsu, *J. Comput. Phys.* **230**, 2031 (2011).
- ³¹⁴H. Jiang, T. Lu, and W. Cai, *J. Comput. Phys.* **258**, 773 (2014).
- ³¹⁵D. Querlioz, H. N. Nguyen, J. Saint-Martin, A. Bourmel, S. Galdin-Retailleau, and P. Dollfus, *J. Comput. Electron.* **8**, 324 (2009).
- ³¹⁶M. D. Croitoru, V. N. Gladilin, V. M. Fomin, J. T. Devreese, W. Magnus, W. Schoenmaker, and B. Sorée, *Solid State Commun.* **147**, 31 (2008).
- ³¹⁷H. Jiang and W. Cai, *J. Comput. Phys.* **229**, 4461 (2010).

- ³¹⁸B.-H. Lee, J. Hur, M.-H. Kang, T. Bang, D.-C. Ahn, D. Lee, K.-H. Kim, and Y.-K. Choi, *Nano Lett.* **16**, 1840 (2016).
- ³¹⁹G. Otnes and M. T. Borgström, *Nano Today* **12**, 31 (2017).
- ³²⁰O. Sheng, C. Jin, J. Luo, H. Yuan, H. Huang, Y. Gan, J. Zhang, Y. Xia, C. Liang, W. Zhang, and X. Tao, *Nano Lett.* **18**, 3104 (2018).
- ³²¹T. Huff, H. Labidi, M. Rashidi, R. Achal, L. Livadaru, T. Dienel, J. Pitters, and R. A. Wolkow, e-print [arXiv:1706.07427](https://arxiv.org/abs/1706.07427).
- ³²²O. Morandi and F. Schürer, *J. Phys. A: Math. Theor.* **44**, 265301 (2011).
- ³²³D. K. Ferry, *Superlattices Microstruct.* **27**, 61 (2000).
- ³²⁴N. Zamponi and L. Barletti, *Math. Methods Appl. Sci.* **34**, 807 (2011).
- ³²⁵H. Tsuchiya, M. Ogawa, and T. Miyoshi, *IEEE Trans. Electron Devices* **39**, 2465 (1992).
- ³²⁶Y. Isawa and T. Hatano, *J. Phys. Soc. Jpn.* **60**, 3108 (1991).
- ³²⁷M. Nedjalkov, D. Vasileska, D. K. Ferry, C. Jacoboni, C. Ringhofer, I. Dimov, and V. Palankovski, *Phys. Rev. B: Condens. Matter Mater. Phys.* **74**, 035311 (2006).
- ³²⁸S. Barraud, *J. Appl. Phys.* **110**, 093710 (2011).
- ³²⁹J. H. Lee, W. J. Jeong, J. Seo, and M. Shin, *Solid-State Electron.* **139**, 101–108 (2018).
- ³³⁰D. Wigger, H. Gehring, V. M. Axt, D. E. Reiter, and T. Kuhn, *J. Comput. Electron.* **15**, 1158 (2016).
- ³³¹P. Weetman and M. S. Wartak, *Phys. Rev. B* **76**, 035332 (2007).
- ³³²H. Linke, W. Sheng, A. Löfgren, H. Xu, P. Omling, and P. E. Lindelof, *Europhys. Lett.* **44**, 341 (1998).
- ³³³D. Zueco and J. L. García-Palacios, *Phys. E* **29**, 435 (2005).
- ³³⁴L. Cleary, W. T. Coffey, Y. P. Kalmykov, and S. V. Titov, *Phys. Rev. E* **80**, 051106 (2009).
- ³³⁵P. Ellinghaus, J. Weinbub, M. Nedjalkov, and S. Selberherr, *Phys. Status Solidi RRL* **11**, 1700102 (2017).
- ³³⁶J. Weinbub, M. Ballicchia, and M. Nedjalkov, *Phys. Status Solidi RRL* **12**, 1800111 (2018).
- ³³⁷J. H. Lee and M. Shin, *J. Comput. Theor. Nanosci.* **14**, 1329 (2017).
- ³³⁸U. Kaczor, B. Klimas, D. Szydłowski, M. Wołoszyn, and B. J. Spisak, *Open Phys.* **14**, 354 (2016).
- ³³⁹W. Heitler and F. London, *Z. Phys.* **44**, 455 (1927).
- ³⁴⁰P. W. Atkins and R. Friedman, *Molecular Quantum Mechanics* (Oxford University Press, 2010).
- ³⁴¹R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, 1965).
- ³⁴²J. A. Poulsen, G. Nyman, and P. J. Rossky, *J. Chem. Phys.* **119**, 12179 (2003).
- ³⁴³K. K. G. Smith, J. A. Poulsen, G. Nyman, and P. J. Rossky, *J. Chem. Phys.* **142**, 244112 (2015).
- ³⁴⁴K. K. G. Smith, J. A. Poulsen, G. Nyman, A. Cunsolo, and P. J. Rossky, *J. Chem. Phys.* **142**, 244113 (2015).
- ³⁴⁵A. Montoya-castillo and D. R. Reichman, *J. Chem. Phys.* **146**, 024107 (2017).
- ³⁴⁶T. J. H. Hele, M. J. Willatt, A. Muolo, and S. C. Althorpe, *J. Chem. Phys.* **142**, 134103 (2015).
- ³⁴⁷Y. Tanimura, *J. Chem. Phys.* **142**, 144110 (2015).
- ³⁴⁸S. Sala, M. Giammarchi, and S. Olivares, *Phys. Rev. A* **94**, 033625 (2016).
- ³⁴⁹M. Barbatti and K. Sen, *Int. J. Quantum Chem.* **116**, 762 (2016).
- ³⁵⁰S. Mai, M. Pollum, L. Martínez-Fernández, N. Dunn, P. Marquetand, I. Corral, C. E. Crespo-Hernández, and L. González, *Nat. Commun.* **7**, 13077 (2016).
- ³⁵¹M. Ruckebauer, S. Mai, P. Marquetand, and L. González, *Sci. Rep.* **6**, 35522 (2016).
- ³⁵²W. Xie, W. Domcke, S. C. Farantos, and S. Y. Grebenshchikov, *J. Chem. Phys.* **144**, 104105 (2016).
- ³⁵³I. A. Ivanov and K. T. Kim, *Phys. Rev. A* **92**, 053418 (2015).
- ³⁵⁴L. Torlina, F. Morales, J. Kaushal, I. Ivanov, A. Kheifets, A. Zielinski, A. Scrinzi, H. G. Muller, S. Sukiasyan, M. Ivanov, and O. Smirnova, *Nat. Phys.* **11**, 503 (2015).
- ³⁵⁵F. E. Low and P. F. Mende, *Ann. Phys. (N. Y.)* **210**, 380 (1991).
- ³⁵⁶T. E. Hartman, *J. Appl. Phys.* **33**, 3427 (1962).
- ³⁵⁷J. Petersen and E. Pollak, *J. Phys. Chem. Lett.* **8**, 4017 (2017).
- ³⁵⁸C. Xu, L. Yu, C. Zhu, J. Yu, and Z. Cao, *Sci. Rep.* **6**, 26768 (2016).
- ³⁵⁹S. Chandrasekhar, *Radiative Transfer* (Dover, 1960).
- ³⁶⁰J. Ville, *Cables Transm.* **2A**, 61 (1948), see <https://archive.org/details/VilleSigAnalytiqueCablesErTrans1948Fr/page/n1>.
- ³⁶¹T. A. C. M. Claassen and W. F. G. Mecklenbräuker, *Philips J. Res.* **35**, 217 (1980), see <https://pdfs.semanticscholar.org/fba5/4e13a54083cdeb76e276d82c3fcbdb475ea96.pdf>.
- ³⁶²T. A. C. M. Claassen and W. F. G. Mecklenbräuker, *Philips J. Res.* **35**, 276 (1980), see <https://pdfs.semanticscholar.org/aaaa/54b49fd67a483a5547b54aa96426db6a3e1d.pdf>.
- ³⁶³L. Cohen, *Time-Frequency Analysis* (Prentice-Hall, 1995).
- ³⁶⁴L. Cohen, *Proc. IEEE* **77**, 941 (1989).
- ³⁶⁵L. Borcea and J. Garnier, *Wave Motion* **63**, 179 (2016).
- ³⁶⁶J. S. Ben-Benjamin, L. Cohen, and P. J. Loughlin, *J. Acoust. Soc. Am.* **138**, 1122 (2015).
- ³⁶⁷A. Marica and E. Zuazua, *Found. Comput. Math.* **15**, 1571 (2015).
- ³⁶⁸J. G. Hoskins and J. C. Schotland, *Phys. Rev. E* **95**, 033002 (2017).
- ³⁶⁹G. Gradoni, S. C. Creagh, G. Tanner, C. Smartt, and D. W. P. Thomas, *New J. Phys.* **17**, 093027 (2015).
- ³⁷⁰S. C. Creagh, G. Gradoni, T. Hartmann, and G. Tanner, *J. Phys. A: Math. Theor.* **50**, 045101 (2017).
- ³⁷¹Y. Wen, Y. Chen, Y. Zhang, H. Chen, and S. Yu, *Phys. Rev. A* **95**, 023825 (2017).
- ³⁷²G. Siviloglou, J. Broky, A. Dogariu, and A. N. Christodoulides, *Phys. Rev. Lett.* **99**, 213901 (2007).
- ³⁷³H.-Y. Liu, E. Jonas, L. Tian, J. Zhong, B. Recht, and L. Waller, *Opt. Express* **23**, 14461 (2015).
- ³⁷⁴Z.-K. Gao, S.-S. Zhang, Q. Cai, Y.-X. Yang, and N.-D. Jin, *Sci. Rep.* **6**, 28151 (2016).
- ³⁷⁵A. Ahrabian, D. Looney, L. Stanković, and D. P. Mandic, *Signal Process.* **106**, 331 (2015).
- ³⁷⁶C. Xu, C. Wang, and W. Liu, *J. Vib. Acoust.* **138**, 051009 (2016).
- ³⁷⁷R. B. Pachori and A. Nishad, *Signal Process.* **120**, 288 (2016).
- ³⁷⁸J. R. De Oliveira Neto and J. B. Lima, *IEEE Trans. Signal Process.* **65**, 6171 (2017).