Proceed. Int. AMSE Conf. "Modelling & Simulation", Sorrento (Italy)
Sept. 29–Oct. 1, 1986, Vol. 2.1, p. 109-120

A Green's Function Approach to the Electrostatic Problem of Single, Coupled and Comb-like Metallic Structures in Anisotropic Multilayered Media

A. R. Baghai-Wadji, S. Selberherr and F. Seifert

AUSTRIA

Abstract- Applying the concept of spectral domain representation, the moment method or the point-matching procedure and Green's function theory, a unified approach for the computation of the static capacitance of single, coupled and comb-like metallic structures in anisotropic multilayered media is presented. The media can be electrically shielded on one or both sides. The geometry of the parallel strips may be arbitrary and not all the strips need to be driven electrically, i.e. they may float. An overview of the relevant methods is followed by a discussion of an efficient approach with a wide range of applicabilty.

I.Introduction

Many problems of greatest practical interest demand an efficient solution procedure for the calculation of the capacitance for metallic strips, deposited on the surface or within a layered dielectric structure [1]. Because of the complexity of the associated boundary value problem and due to the fact that generally the layers are dielectrically anisotropic, direct numerical method are highly time-consuming. On the other hand approximate formulae, which are used by other authors have a restricted range of validity. We present a rigorous semi-numerical method of analysis covering a large class of problems. Although only a restricted number of problems is considered here, the method is quite general and can be extended to embrace a class of much more complicated problems. The method of analysis, a uniformly valid representation, is based on the concept of spectral domain representation and the method of moments, which are combined with an auxiliary quantity, the Green's function. First, closed-form formulae for the Green's function in the wavenumber domain have been derived for a variety of cases, which are of practical interest. Using the derived Green's function and the method of moments, the associated integral equation is replaced by a matrix equation, which can easily be solved by standard routines. Although the Green's function theory is by no means a new field, very recently it has found some important applications [2]. The aim of this paper is to show that the advantages of the Green's function can be extended using the method of moments or a specialization of it, called point-matching or collocation method. Using the Fourier transform technique the boundary conditions are transformed into a set of algebraic equations. Therefrom a relation between the spectral components of the potential and charge distribution is derived , which immediately yields a closed-form expression for the Fourier transform of the Green's function. The unknown function, the charge distribution on the metallic strips is expressed in terms of basis or expansion functions. A set of weighting or testing functions and an appropriately defined innerproduct transform the original functional equation into a matrix equation. The elements of the resulting matrix are generally integrals, which must be evaluated numerically. However, for some cases of greatest practical interest they can be solved analytically. The inversion of the obtained matrix yields the unknown charge values on the metallic strips, and their capacitance consecutively.

II.Theory

Assume a sandwich structure consisting of l layers, ($l+1$ interfaces), as sketched in Fig.1. Each layer represents a dielectric medium which can be anisotropic, isotropic or the free space. To unify the description of the method, the two halfspaces at the top and the bottom of the configuration considered are also regarded as layers.

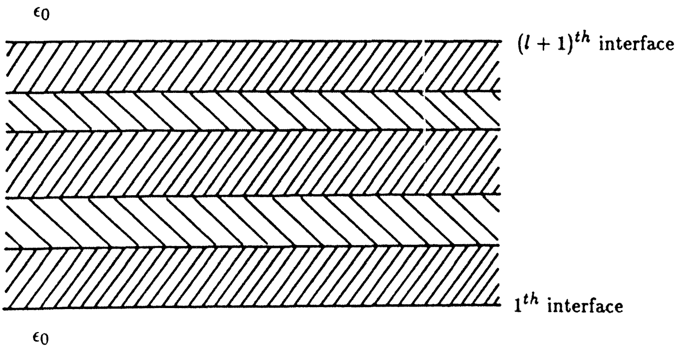


Fig.1 A dielectric sandwich structure

On one of the interfaces we assume N parallel thin metallic strips. The strips may have ideal conductivity. The strips are infinite (there is no variation in x- direction, $(\frac{\partial}{\partial x} \equiv 0)$, a restriction which is not conceptually necessary and can be removed. The geometry (width and spacing) of the strips may be arbitrary and not all the strips must be driven electrically, i.e. they can float. The problem is to find the charge distribution on the fingers (capacitances).

Further, in this paper we restrict ourselves to structures, in which the electrically active zone (metallic strips) only lies on one of the interfaces P, Fig.2. For this class of problems the Green's function is a scalar function.

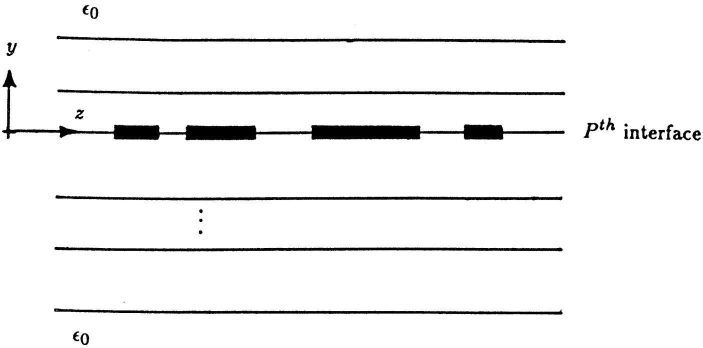


Fig.2 Geometry of interest

For problems in which the active zone is split into k parallel planes, Fig.3, the Green's function is a matrix function of rank k. This class of problems will be discussed elsewhere.

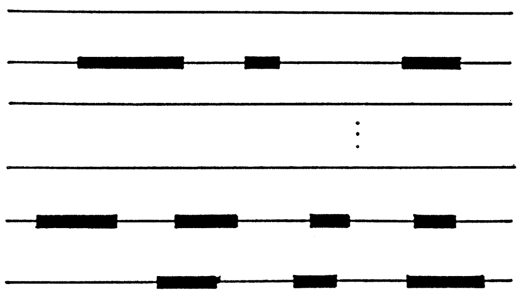


Fig.3 A sandwich structure with more than one active rigion

As we will see, the main task is to find an expression for the Green's function of the boundary value problem of interest. Before we describe the formalism, let us say a little more about the Green's function and its relevant properties. Consider again the problem sketched in Fig.2. On the interface P, instead of the metallic strips, a line charge source may excite the medium. The potential response of the medium to the line charge source is called the Green's function for the boundary value problem, $\overline{G}(k_z)$. Due to the fact that a line charge source can be isolated or electrically shielded, two different cases for $\overline{G}(k_z)$ must be distinguished:

i) For an isolated line charge source, $\overline{G}(k_z)$ has a pole singularity. The reason is that an isolated line charge source cannot exist in reality. As one can easily show, the charge neutrality condition for the whole metallic structure removes the pole singularity and makes the occuring formulae regular.

ii) For a non-isolated line charge source the corresponding Green's function $\overline{G}(k_z)$ is regular in itself. These aspects will be illustrated by some examples, after we have discussed the basic formulae.

On P the spatial distribution of the potential $\phi(z)$ can be written as

$$\phi(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{\phi}(k_z) e^{-jk_z z} dk_z + C \qquad (II.1)$$

where the bar denotes Fourier transformation. The constant parameter C is discussed in [4]. Our main task is to find an expression for $\overline{\phi}(k_z)$ as follows:
The potential and the charge distribution $\rho(z)$ on P are related by

$$\phi(z) = \int_{-\infty}^{+\infty} G(z\prime - z)\rho(z\prime)dz\prime + C \qquad (II.2)$$

or equivalently in the wavenumber space by

$$\overline{\phi}(k_z) = \overline{G}(k_z)\overline{\rho}(k_z) \qquad (II.3)$$

where $G(z)$ and $\overline{G}(k_z)$ denote Green's functions in the spatial and in the wavenumber domain, respectively. Insertion of (II.3) in (II.1) yields

$$\phi(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{G}(k_z)\overline{\rho}(k_z) e^{-jk_z z} dk_z + C \qquad (II.4)$$

Notice that (II.4) relates the potential distribution on P to the Fourier transformation of the charge density on P. Generally the potential values of the strips are given, and the problem is to find the charge distribution on the strips. One can show that [3], [4], once the charge distribution on the strips has been calculated, the field problem is uniquely determined.

III. Approximation of the charge density

To obtain an approximate formula for $\overline{\rho}(k_z)$ in (II.4), first the fingers have to be appropriately discretized into substrips, Fig.4

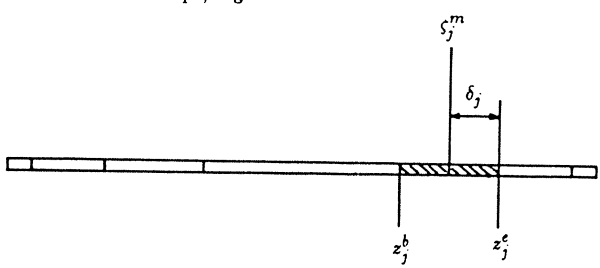


Fig.4 Discretization of the metallic strips

Fig.4 shows a nonequidistant discretization, which takes into account the charge singularities at finger-edges. Using the stepfunction approximation for the charge density

$$\rho(z) = \rho_0 \sum_{j=1}^{N} \sigma_j \cdot [H(z - z_j^b) - H(z - z_j^e)] \tag{III.5}$$

and performing the Fourier transformation, we obtain

$$\overline{\rho}(k_z) = \rho_0 \sum_{j=1}^{N} \sigma_j \frac{e^{jk_z z_j^e} - e^{jk_z z_j^b}}{jk_z} \tag{III.6}$$

where $H(z)$ and ρ_0 are the Heaviside stepfunction and the charge normalization factor, respectively. Inserting (III.6) in (II.4) and interchanging the order of summation and integration, we have

$$\phi(z) = \frac{\rho_0}{2\pi\epsilon_0} \sum_{j=1}^{N} \sigma_j \int_{-\infty}^{+\infty} \epsilon_0 \overline{G}(k_z) \frac{1}{jk_z} [e^{jk_z(z_j^e - z)} - e^{jk_z(z_j^b - z)}] dk_z + C \tag{III.7}$$

or upon $\frac{\rho_0}{2\pi\epsilon_0} = 1$

$$\phi(z) = \sum_{j=1}^{N} \sigma_j \delta_j \cdot \frac{1}{\delta_j} \int_{-\infty}^{+\infty} \frac{\epsilon_0 \overline{G}(k_z)}{jk_z} [e^{jk_z(z_j^e - z)} - e^{jk_z(z_j^b - z)}] dk_z + C \tag{III.8}$$

Notice that the formulae in this and in the following two sections are generally valid and independent of the specific configuration of the layered structure and of the constituted properties of the dielectrics used. These aspects manifest themselves in the special forms of the corresponding Green's functions, a fact which will be demonstrated below by characteristic samples.

IV.Point-matching

For the moment let us assume that all the strips are electrically driven (there are no floating strips). In this case, denoting the potential and the midpoint coordinate of the i^{th} substrip by ϕ_i and ς_i^m, respectively, we have

$$\phi(\varsigma_i^m) = \sum_{j=1}^{N} \sigma_j \delta_j \cdot \frac{1}{\delta_j} \int_{-\infty}^{+\infty} \frac{\epsilon_0 \overline{G}(k_z)}{jk_z} [e^{jk_z(z_j^e - \varsigma_i^m)} - e^{jk_z(z_j^b - \varsigma_i^m)}] dk_z + C \tag{IV.9}$$

or in a more compact form

$$\phi_i = \sum_{j=1}^{N} \sigma_j \delta_j \cdot A(i,j) + C, \quad i = 1 \,..\, N \tag{IV.10-a}$$

where

$$A(i,j) = \frac{2}{\delta_j} \int_{0}^{+\infty} \frac{\epsilon_0 \overline{G}(k_z)}{k_z} [sink_z(z_j^e - \varsigma_i^m) - sink_z(z_j^b - \varsigma_i^m)] dk_z \tag{IV.10-b}$$

Although (IV.10-b) is quite general, there is a lack of symmetry with respect to an interchange of the indices i and j, $A(i,j) \neq A(j,i)$, a deficiency which will be eliminated in the next section.

V.Method of moments

The nonsymmetry property $A(i,j) \neq A(j,i)$ is an inherent attribute of the point-matching procedure, which generally occurs in similar problems, if we use non-equidistant discretization. On the other hand, to take into account the charge singularity at finger-edges and to reduce computer resources (memory and time), the metallic strips have to be discretized non-equidistantly. Applying the method of moments, below it is shown that $A(i,j) = A(j,i)$, independent of the discretization, equidistant or non-equidistant. For simplicity we write again the approximate formula for the $\phi(z)$

$$\phi(z) = \sum_{j=1}^{N} \sigma_j \delta_j \cdot \frac{1}{\delta_j} \int_{-\infty}^{+\infty} \frac{\epsilon_0 \overline{G}(k_z)}{jk_z} [e^{jk_z(z_j^e - z)} - e^{jk_z(z_j^b - z)}] dk_z + C \qquad (V.1)$$

Now we use the mean value of the potential on i^{th} substrip for the ϕ_i,

$$\phi_i = \frac{1}{z_i^e - z_i^b} \int_{z_i^b}^{z_i^e} \phi(z) dz \qquad (V.2)$$

Inserting (V.1) in (V.2), interchanging the order of summation and integration

$$\int_{z_i^b}^{z_i^e} \sum_{j=1}^{N} \longrightarrow \sum_{j=1}^{N} \int_{z_i^b}^{z_i^e}$$

and finally performing the integration $\int_{z_i^b}^{z_i^e} .. \, dz$ we obtain

$$\phi_i = \sum_{j=1}^{N} \sigma_j \delta_j \cdot \frac{1}{(z_i^e - z_i^b)(z_j^e - z_j^b)} \int_{-\infty}^{+\infty} \frac{2\epsilon_0 \overline{G}(k_z)}{k_z^2} \cdot$$
$$\cdot [e^{jk_z(z_j^e - z_i^e)} - e^{jk_z(z_j^e - z_i^b)} - e^{jk_z(z_j^b - z_i^e)} + e^{jk_z(z_j^b - z_i^b)}] dk_z + C \qquad (V.3)$$

or more compactly

$$\phi_i = \sum_{j=1}^{N} \sigma_j \delta_j \cdot A(i,j) + C \qquad (V.4)$$

with

$$A(i,j) = \frac{1}{(z_i^e - z_i^b)(z_j^e - z_j^b)} \int_{0}^{+\infty} \frac{4\epsilon_0 \overline{G}(k_z)}{k_z^2} \cdot$$
$$\cdot [\cos k_z(z_j^e - z_i^e) - \cos k_z(z_j^e - z_i^b) - \cos k_z(z_j^b - z_i^e) + \cos k_z(z_j^b - z_i^b)] dk_z$$

$$(V.5)$$

VI.Calculation of the Green's function in the wavenumber domain

Assume an unbounded anisotropic dielectric, which is fully characterized by a $3x3$ symmetric positive definite matrix $(\underline{\epsilon})$. Further, assuming that there is no spatial variation in $x-$direction ($\frac{\partial}{\partial x} \equiv 0$), the distribution of the electrical potential, $\phi(y,z)$, is governed by the Laplace equation

$$\epsilon_{22}\frac{\partial^2\phi}{\partial y^2} + 2\cdot\epsilon_{23}\frac{\partial^2\phi}{\partial y\partial z} + \epsilon_{33}\frac{\partial^2\phi}{\partial z^2} = 0 \qquad (VI.1)$$

The $y-$component of the electric displacement vector $\vec{D}$ is

$$D_y = -\epsilon_{22}\frac{\partial\phi}{\partial y} - \epsilon_{23}\frac{\partial\phi}{\partial z} \qquad (VI.2)$$

Defining ϵ_P by

$$\epsilon_P = \sqrt{\epsilon_{22}\epsilon_{33} - \epsilon_{23}^2}$$

let us now schematically discuss the solution procedure for the calculation of the Green's function in the wavenumber domain.

The complexity of the considered boundary value problem is directly reflected in the form of the Green's function. However, one can show that for a certain class of problems (halfspace, multilayered structures) the Green's function has the general form

$$\overline{G}(k_z) = \frac{1}{\epsilon_0|k_z|}\cdot f(|k_z|) = \overline{G}(|k_z|)$$

wherein the function $f(|k_z|)$ may or may not have a pole singularity in $k_z = 0$, depending on if $\overline{G}(k_z)$ is resulted from an isolated or a nonisolated line charge source, as we will see in more detail below.

Now let us find general solutions for the Laplace equation (VI.1) for the following three fundamental cases, from which every multilayered structure can be build up.

Solution for the Laplace's equation

i) $a < y < b$, a and b finite, z arbitrary Fig.5

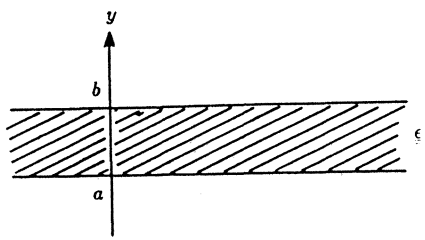


Fig.5 A dielectric medium bounded by lower and upper sides

$$\phi(y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[A(k_z)cosh(\frac{\epsilon_P}{\epsilon_{22}}|k_z|y)+$$
$$+B(k_z)sinh(\frac{\epsilon_P}{\epsilon_{22}}|k_z|y)]\cdot e^{j\frac{\epsilon_{23}}{\epsilon_{22}}k_z y}\cdot e^{-jk_z z}dk_z + C \qquad (VI.3)$$

$$D_y(y,z) = \frac{-\epsilon_0\epsilon_P}{2\pi}\int_{-\infty}^{+\infty}|k_z|\cdot[A(k_z)sinh(\frac{\epsilon_P}{\epsilon_{22}}|k_z|y)+$$
$$+B(k_z)cosh(\frac{\epsilon_P}{\epsilon_{22}}|k_z|y)]\cdot e^{j\frac{\epsilon_{23}}{\epsilon_{22}}k_z y}\cdot e^{-jk_z z}dk_z \qquad (VI.4)$$

ii) $a < y < +\infty$, a finite, z arbitrary, Fig.6

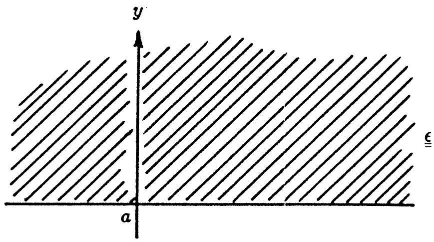


Fig.6 A dielectric medium bounded by lower side

$$\phi(y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}A(k_z)\cdot e^{-\frac{\epsilon_P}{\epsilon_{22}}|k_z|y}\cdot e^{j\frac{\epsilon_{23}}{\epsilon_{22}}k_z y}\cdot e^{-jk_z z}dk_z + C \qquad (VI.5)$$

$$D_y(y,z) = \frac{\epsilon_0\epsilon_P}{2\pi}\int_{-\infty}^{+\infty}|k_z|A(k_z)\cdot e^{-\frac{\epsilon_P}{\epsilon_{22}}|k_z|y}\cdot e^{j\frac{\epsilon_{23}}{\epsilon_{22}}k_z y}\cdot e^{-jk_z z}dk_z \qquad (VI.6)$$

Note that (VI.5) and (VI.6) respectively, can be deduced from (VI.3) and (VI.4) by $B(k_z) = -A(k_z)$.

iii) $-\infty < y < b$, b finite, z arbitrary, Fig.7

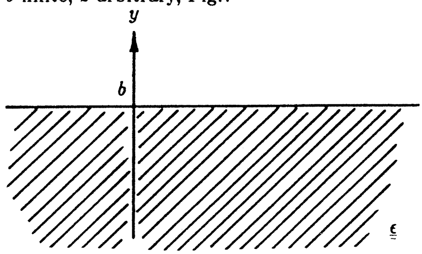


Fig.7 A dielectric medium bounded by upper side

$$\phi(y,z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(k_z) \cdot e^{\frac{\epsilon_P}{\epsilon_{22}}|k_z|y} \cdot e^{j\frac{\epsilon_{23}}{\epsilon_{22}}k_z y} \cdot e^{-jk_z z} dk_z + C \qquad (VI.7)$$

$$D_y(y,z) = \frac{\epsilon_0 \epsilon_P}{2\pi} \int_{-\infty}^{+\infty} |k_z| A(k_z) \cdot e^{\frac{\epsilon_P}{\epsilon_{22}}|k_z|y} \cdot e^{j\frac{\epsilon_{23}}{\epsilon_{22}}k_z y} \cdot e^{-jk_z z} dk_z \qquad (VI.8)$$

Similarly (VI.7) and (VI.8) can be deduced from (VI.3) and (VI.4), respectively, by the constraint $B(k_z) = A(k_z)$.

Boundary conditions

Let us concentrate on an interface Q, which is with a distance y_1 parallel to the (x,z)-plane, Fig.8

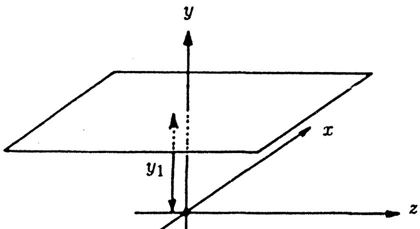


Fig.8 l^{th} interface

The following boundary conditions must be met:

i)

$$\phi(y_1^+, z) = \phi(y_1^-, z) \qquad (VI.9)$$

ii-a) if Q does not coinside with P

$$D_y(y_1^+, z) - D_y(y_1^-, z) = 0 \qquad (VI.10)$$

ii-b) if Q coinsides with P

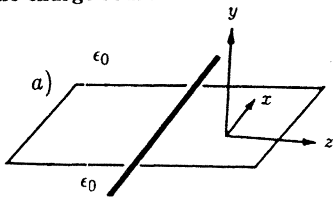
$$D_y(y_1^+, z) - D_y(y_1^-, z) = \rho(z) \qquad (VI.11)$$

With (VI.3)-(VI.11) a given multilayered structure can now be analyzed as follows: For the l^{th}layer, depending on the condition $a < y < b$, $a < y < +\infty$ or $-\infty < y < b$, use the corresponding formulae for $\phi(y,z)$ and $D_y(y,z)$ with $A_l(k_z)$ and $B_l(k_z)$ as unknowns. With regard to the boundary conditions stated above, establish then a system of algebraical equations for $A_l(k_z)$ and $B_l(k_z)$. Finally, succesively eliminating the unknowns,results in the Green's function in the wavenumber space, $\overline{G}(k_z)$.

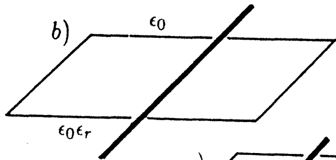
VII.Examples

Below some simple boundary value problems with the associated Green's functions are given, where we have used the algorithim discussed in the preceding sections.

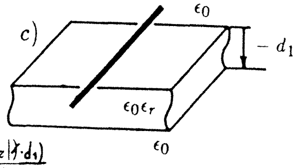
Isolated line charge source



$$\overline{G}_a(k_z) = \overline{G}_a(|k_z|) = \frac{1}{\epsilon_0|k_z|} \cdot \frac{1}{1+1}$$



$$\overline{G}_b(k_z) = \overline{G}_b(|k_z|) = \frac{1}{\epsilon_0|k_z|} \cdot \frac{1}{1+\epsilon_r}$$



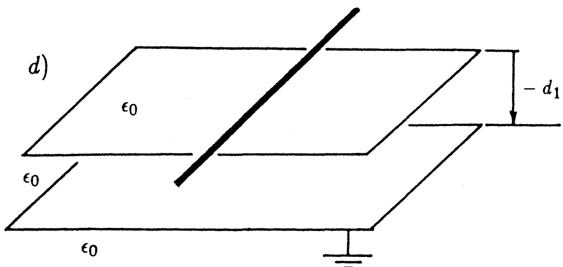
$$\overline{G}_c(k_z) = \overline{G}_c(|k_z|) = \frac{1}{\epsilon_0|k_z|} \cdot \frac{1}{1+\epsilon_r \cdot \overline{g}_c(|k_z|)}$$

where

$$\overline{g}_c(|k_z|) = \frac{1 + \epsilon_r coth(|k_z|\gamma \cdot d_1)}{\epsilon_r + coth(|k_z|\gamma \cdot d_1)}$$

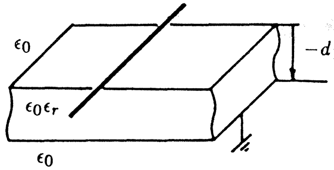
Non-isolated line charge source

$$\overline{G}_d(k_z) = \overline{G}_d(|k_z|) = \frac{1}{\epsilon_0|k_z|} \cdot \frac{1}{1+coth(d_1|k_z|)}$$



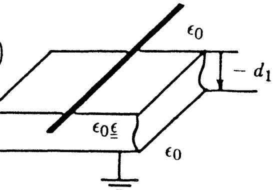
$$\overline{G}_e(k_z) = \overline{G}_e(|k_z|) = \frac{1}{\epsilon_0|k_z|}\cdot\frac{1}{1+\epsilon_r coth(d_1|k_z|)}$$

e)



$$\overline{G}_f(k_z) = \overline{G}_f(|k_z|) = \frac{1}{\epsilon_0|k_z|}\cdot\frac{1}{\epsilon_{rP} coth(\frac{\epsilon_{rP}}{\epsilon_{22}} d_1|k_z|)+1}$$

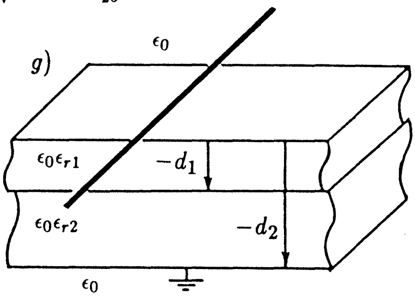
f)



where

$$\epsilon_{rP} = \sqrt{\epsilon_{22}\epsilon_{33} - \epsilon_{23}^2}$$

g)



$$\overline{G}_g(k_z) = \overline{G}_g(|k_z|) = \frac{1}{\epsilon_0|k_z|}\cdot\frac{1}{1+\epsilon_{r1}\bar{g}_g(|k_z|)}$$

where

$$\bar{g}_g(|k_z|) = \frac{cosh(d_2|k_z|) + \epsilon_{r12}cosh[(2d_1 - d_2)|k_z|]}{sihh(d_2|k_z|) + \epsilon_{r12}sinh[(2d_1 - d_2)|k_z|]}$$

and

$$\epsilon_{r12} = \frac{\epsilon_{r2} - \epsilon_{r1}}{\epsilon_{r2} + \epsilon_{r1}}$$

Notice that

i) $\overline{G}(k_z) = \overline{G}(|k_z|)$

$\therefore$ the Green's functions have the general form

$$\overline{G}(k_z) = \frac{1}{\epsilon_0|k_z|}\cdot\overline{f}(|k_z|)$$

iii) in the limit $k_z \rightarrow 0$, $\overline{f}(|k_z|)$ tends to zero or to a constant value, depending on the nature of the problem.

For an isolated line charge source

$$\overline{f}(|k_z|) \rightarrow const, \quad k_z \rightarrow 0$$

whereas for a nonisolated line charge source

$$\overline{f}(|k_z|) \rightarrow 0, \quad k_z \rightarrow 0$$

Conclusion

Using Fourier transformation and Green's function theory combined with the point-matching procedure or the method of moments a unified solution concept for multi-layered dielectric media has been presented. The formulae obtained by applying the point-matching procedure and the method of moments, respectively, have been compared. Different properties of Green's functions for an isolated and a non-isolated line charge source are discussed.

Acknowledgments

This work was supported by Siemens AG, Central Research Laboratories, Munich, West Germany. Partial support by the Austrian Science Research Fund Project 5311 is acknowledged. Valuable comments and discussions with Doz.M.Kowatsch and the continuous encouragement by Dr.H.Stocker are gratefully acknowledged.

References

[1] N.G.Alexopoulos, Integrated-circuit structures on anisotropic substrates, IEEE Trans. Microwave Theory and Techn., 1985, vol. MTT-33, No. 10, oct 1986, pp.847-881

[2] R.F.Milsom, N.H.C.Reilly and M.Redwood, Analysis of generation and detection of surface and bulk acoustic waves by interdigital transducers, IEEE Trans., SU-24, pp.147-166, 1977

[3] A.R.Baghai-Wadji, S.Selberherr and F.J.Seifert, On the calculation of charge, electrostatic potential and capacitance in generalized finite SAW structures, Ultrasonics Symp. proceedings, pp.44-48, 1984

[4] A.R.Baghai-Wadji, S.Selberherr and F.J.Seifert, Closed-form electrostatic field analysis for metallic comb-like structures containing single and interconnected floating strips of arbitrary topological complexity, part I:One-dimensional representation, part II:Two-dimensional representation, will be published in 2^{nd} International IGTE Symp. proceedings, Graz, Austria, 1986