A Finite Element Time-Domain Algorithm Based on the Alternating-Direction Implicit Method

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***Abstract*— We present an implicit finite element time-domain (FETD) solution of the Maxwell equations. The time-dependent formulation employes a time-integration method based on the alternating-direction implicit (ADI) method. The ADI method is directly applied to the Maxwell equations in order to obtain an unconditionally stable FETD approach. The method uses edge elements for electric field and facet elements for magnetic flux density. The advantage of this formulation based on the ADI scheme is that the time step is no longer governed by the spatial discretization of the mesh, but rather by the spectral content of the time-dependent signal. Numerical formulations are presented and simulation results are compared to those using the conventional FETD method.**

***Index Terms*— Alternating-direction implicit (ADI) method, finite element time-domain (FETD) method, Maxwell's equations.**

I. Introduction

In the past few years, numerical methods have been extensively used to solve the Maxwell equations in the time domain for the analysis of transient problems. Moreover, it is often important to analyze systems in a large frequency band. Time-domain methods are well suited to achieve this target, because they can obtain broadband information from a single computation. In electromagnetic compatibility (EMC) studies or antenna analysis, for instance, we can obtain the response of a system over a wide frequency band by using a large-band transient signal as excitation and by applying the Fourier transform on the results. This can give a significant reduction of the computational cost compared to frequency domain calculations.

Several methods can be used to calculate the time domain solution of electromagnetic problems. The most popular one is the finite difference time-domain (FDTD) algorithm, introduced by Yee in 1966 [1]. The FDTD method discretizes the time-dependent Maxwell curl equations using central differences in time and space and a leap-frog explicit scheme for time integration. Its principal advantage is ease of implementation. However, in its conventional form, the grid for the spatial discretization is Cartesian and uniform in nature. Consequently, the FDTD method restricts geometry representation to stair-stepped-shaped boundaries, which results in a large burden on the memory resources and the CPU time, when the method is applied to geometries with curvature and/or fine features. Finite element time-domain (FETD) methods combine the advantages of a time-domain technique and the versatility of their spatial discretization procedures [2]. They allow accurate modeling of complex structures with arbitrary shaped regions and take easily inhomogeneous materials into account.

The FETD approaches developed so far can be grouped into two classes. One class of approaches directly solves Maxwell's equations and operates in a leap-frog fashion similar to the FDTD method (explicit method). These approaches are conditionally stable [3], [4]. Another class of FETD approaches use the second-order vector wave equation, or the curl-curl equation, obtained by eliminating one of the filled variables from Maxwell's equations. These solvers can be formulated to be unconditionally stable [5], [6] or conditionally stable [7], [8]. In an unconditionally stable scheme the time step is not constrained by a stability criterion. However, it is limited by the accuracy requirement and also by the spectral content of temporal signatures. Therefore, if the minimum cell size in the computational domain is required to be much smaller than the wavelength, these schemes are more efficient in terms of computer resources such as CPU time.

Recently, a new method called the alternating direction implicit finite-difference time-domain (ADI-FDTD) method has been introduced to solve Maxwell's curl equations using finite difference discretization [9]. This method is an attractive alternative to the standard FDTD due to its unconditional stability with moderate computational overhead. In this paper a FETD approach based on the ADI technique is presented. This technique is directly applied to the first order Maxwell's equations and leads to an unconditionally stable approach. The ADI technique was first applied to Yee's grid in order to formulate an implicit FDTD scheme [9]. Here, we applied this technique to the FETD method to offer an unconditionally stable finite element time-domain method.

II. ADI Principle

For explanation of the ADI method as a technique for the development of an implicit integration scheme, the time-dependent curl vector equations of Maxwell's equations are considered:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{1}$$

These equations can be cast into six scalar partial differential equations in the Cartesian coordinates. We consider the following scalar equation from the above given system:

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) \tag{2}$$

By applying the ADI principle which is widely used in solving parabolic equations [10], the computation of equation (2) for the FETD solution marching from the nth time step to the $(n+1)$th time step is broken up into two computational subadvancements: the advancement from the nth time step to the $(n+1/2)$th time step and the advancement from the $(n+1/2)$th time step to the $(n+1)$th time step. More specifically, the two substeps are as follows:

1) For the first half-step, *i.e.*, at the $(n+1/2)$th time step, the first partial derivative on the right hand side of (2), $\partial E_x/\partial y$, is replaced with its unknown pivotal values at $(n+1/2)$th time step; while the second partial derivatives on the right hand side, $\partial E_y/\partial x$, is replaced with its known values at the previous nth time step. In other words:

$$\frac{H_z^{n+\frac{1}{2}} - H_z^n}{\Delta t/2} = \frac{1}{\mu}\left(\frac{\partial E_x^{n+\frac{1}{2}}}{\partial y} - \frac{\partial E_y^n}{\partial x}\right) \tag{3}$$

2) For the second half time step, *i.e.*, at $(n+1)$th time step, the second term on the right hand side, $\partial E_y/\partial x$, is replaced with its unknown pivotal values at $(n+1)$th time step; while the first term, $\partial E_x/\partial y$, is replaced with its known values at the previous $(n+1/2)$th time step. In other words:

$$\frac{H_z^{n+1} - H_z^{n+\frac{1}{2}}}{\Delta t/2} = \frac{1}{\mu}\left(\frac{\partial E_x^{n+\frac{1}{2}}}{\partial y} - \frac{\partial E_y^{n+1}}{\partial x}\right) \tag{4}$$

The above two substeps represent the alternations in the FETD recursive computation directions in the sequence of the terms, the *first* and the *second* term. They result in the implicit formulations as the right hand side's of the equations contain the field values unknown and to be updated. The technique is then termed "the alternating direction implicit" technique. Attention should also be paid to the fact that no time-step difference (or lagging) between electric and magnetic field components is present in the formulations.

Applying the same procedure to all of the other five scalar differential equations of Maxwell's equations, one obtains the complete set of the implicit formula.

III. ADI-FETD Method

This section describes the ADI-FETD formulation for analyzing two-dimensional electromagnetic problems. Throughout, all fields are assumed to be TE_z polarized; the proposed scheme, however, also can be applied to TM_z problems with minor modifications. Moreover, the method can be easily extended to three-dimensional problems.

The Maxwell curl equations governing the solution of a two-dimensional problem, TE_z case, *i.e.* $H_x = H_y = E_z = 0$, in a lossless medium are given by:

$$\frac{1}{\mu}\left(\frac{\partial B_z}{\partial t}\right) = -\frac{1}{\mu}\left(\nabla \times \mathbf{E}\right).\hat{\mathbf{z}}$$
$$\epsilon\left(\frac{\partial \mathbf{E}}{\partial t}\right) = \nabla \times \left(\frac{1}{\mu}B_z\hat{\mathbf{z}}\right) - \mathbf{J} \tag{5}$$

where $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}$ is the electric field and $\mathbf{B} = B_z\hat{\mathbf{z}}$ is the magnetic flux density.

The examined two-dimensional domain Ω in the xy-plane is assumed to be discretized by a FE mesh composed by N_t triangular elements, N_e edges and N_f faces. In each point $\mathbf{r}$ of the element, Ω^e, the electric field $\mathbf{E}$ and the magnetic flux density $\mathbf{B}$ are approximated by edge elements and facet elements, respectively, as:

$$\mathbf{E}^e(\mathbf{r},t) = \sum_{i=1}^{3} e_i(t)\mathbf{W}_i(\mathbf{r})$$
$$\mathbf{B}^e(\mathbf{r},t) = \sum_{i=1}^{1} b_i(t)\mathbf{F}_i(\mathbf{r}) \tag{6}$$

where $e_i(t)$ is the electric field circulation along the i-th edge, $b_i(t)$ is the flux of the magnetic flux density through i-th face, $\mathbf{W}_i$ is the Whitney 1-form vector basis function associated to the i-th edge and $\mathbf{F}_i$ is the Whitney 2-form vector basis function associated to the i-th face [11], such that

$$\mathbf{W} \in \mathcal{H}(curl;\Omega) = \{\mathbf{u} : \nabla \times \mathbf{u} \in [\mathcal{L}^2(\Omega)]^3\}$$
$$\mathbf{F} \in \mathcal{H}(div;\Omega) = \{\mathbf{u} : \nabla.\mathbf{u} \in \mathcal{L}^2(\Omega)\} \tag{7}$$

For Whitney 1-forms, the basis functions are well known by now. For example, for $edge\{mn\}$ where m and n are nodes of the edge, it is:

$$\mathbf{W} = \xi_m \nabla \xi_n - \xi_n \nabla \xi_m \tag{8}$$

where ξ_m is the Lagrange interpolation polynomial at vertex m [11]. Similarly, the vector basis functions for Whitney 2-forms associated with a particular $facet\{mnp\}$ where m, n and p are nodes of the face, can be written as:

$$\mathbf{F} = 2\left(\xi_m \nabla \xi_n \times \nabla \xi_p + \xi_n \nabla \xi_p \times \nabla \xi_m + \xi_p \nabla \xi_m \times \nabla \xi_n\right) \tag{9}$$

The Galerkin method is applied to the Maxwell curl equations (5) using the field approximations (6). Using the ADI procedure for time marching, which was explained in the previous section, the following equations are obtained:

1) $(n+\frac{1}{2})$th time step

$$G_z^e \frac{b^{n+\frac{1}{2}} - b^n}{\Delta t/2} = -\left(K_1^e e^{n+\frac{1}{2}} + K_2^e e^n\right)$$
$$(C_x^e + C_y^e)\frac{e^{n+\frac{1}{2}} - e^n}{\Delta t/2} =$$
$$L_1^e b^{n+\frac{1}{2}} + L_2^e b^n - q^{en+\frac{1}{2}} \tag{10}$$

2) $(n+1)$th time step

$$G_z^e \frac{b^{n+1} - b^{n+\frac{1}{2}}}{\Delta t/2} = -\left(K_1^e e^{n+\frac{1}{2}} + K_2^e e^{n+1}\right)$$

$$(C_x^e + C_y^e)\frac{e^{n+1} - e^{n+\frac{1}{2}}}{\Delta t/2} = L_1^e b^{n+\frac{1}{2}} + L_2^e b^{n+1} - q^{e\,n+1} \quad (11)$$

where the matrices are given by:

$$\begin{aligned} G_{zij}^e &= \langle \mathbf{F}_i, \mu^{-1}\mathbf{F}_j \rangle_{\Omega_e} \\ K_{1ij}^e &= \langle \mathbf{F}_i, \mu^{-1}\nabla \times (\mathbf{W}_j.\hat{\mathbf{x}})\,\hat{\mathbf{x}} \rangle_{\Omega_e} \\ K_{2ij}^e &= \langle \mathbf{F}_i, \mu^{-1}\nabla \times (\mathbf{W}_j.\hat{\mathbf{y}})\,\hat{\mathbf{y}} \rangle_{\Omega_e} \\ (C_x^e + C_y^e) &= C_{ij}^e = \langle \mathbf{W}_i, \epsilon \mathbf{W}_j \rangle_{\Omega_e} \\ L_{1ij}^e &= \langle \mathbf{F}_j, \mu^{-1}\nabla \times (\mathbf{W}_i.\hat{\mathbf{x}})\,\hat{\mathbf{x}} \rangle_{\Omega_e} = K_{1ij}^{eT} \\ L_{2ij}^e &= \langle \mathbf{F}_j, \mu^{-1}\nabla \times (\mathbf{W}_i.\hat{\mathbf{y}})\,\hat{\mathbf{y}} \rangle_{\Omega_e} = K_{2ij}^{eT} \\ q_i^e &= \langle \mathbf{W}_i, \mathbf{J} \rangle_{\Omega_e} \end{aligned} \quad (12)$$

By substituting the expressions for $e^{n+\frac{1}{2}}$ and e^{n+1} represented by the second equations of (10) and (11) into their first equations and transfering the local equations to a global system, one can obtain:

1) $(n+\frac{1}{2})$th time step

$$\left(G_z + \frac{\Delta t^2}{4}K_1 C^{-1} L_1\right) b^{n+\frac{1}{2}} = \left(G_z - \frac{\Delta t^2}{4}K_1 C^{-1} L_2\right) b^n - \frac{\Delta t}{2}(K_1 + K_2)e^n + \frac{\Delta t^2}{4}K_1 C^{-1} q^{n+\frac{1}{2}}$$

$$Ce^{n+\frac{1}{2}} = Ce^n + \frac{\Delta t}{2}L_1 b^{n+\frac{1}{2}} + \frac{\Delta t}{2}L_2 b^n - \frac{\Delta t}{2}q^{n+\frac{1}{2}} \quad (13)$$

2) $(n+1)$th time step

$$\left(G_z + \frac{\Delta t^2}{4}K_2 C^{-1} L_1\right) b^{n+1} = \left(G_z - \frac{\Delta t^2}{4}K_2 C^{-1} L_2\right) b^{n+\frac{1}{2}} - \frac{\Delta t}{2}(K_1 + K_2)e^{n+\frac{1}{2}} + \frac{\Delta t^2}{4}K_2 C^{-1} q^{n+1}$$

$$Ce^{n+1} = Ce^{n+\frac{1}{2}} + \frac{\Delta t}{2}L_1 b^{n+\frac{1}{2}} + \frac{\Delta t}{2}L_2 b^{n+1} - \frac{\Delta t}{2}q^{n+1} \quad (14)$$

IV. Numerical Results

To demonstrate the validity of the proposed ADI-FETD method, a two-dimensional rectangular cavity was computed with both the proposed and the conventional FETD method. In the conventional FETD method, for time discretization a leap-frog scheme is applied. The cavity with the dimension of 1 mm×1 mm is modeled using rectangular elements. For both the ADI-FDTD and the conventional FDTD, a similar mesh with 722 elements and 1121 edges was used.

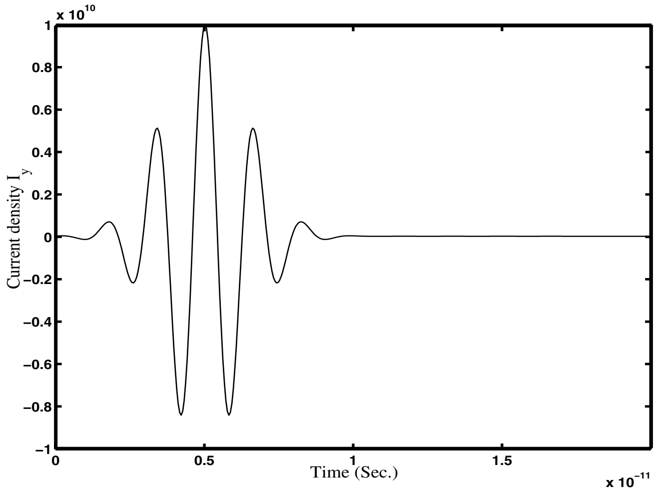


Fig. 1. The time variations of current density, I_y, used as an excitation for the two-dimensional cavity.

The resonant frequencies can be obtained by launching a time signal and applying the Fourier transform on the time response. An excitation sinus modulated Gaussian as current density is used in this simulation. Fig. 1 shows the time variations of this current density used as excitation.

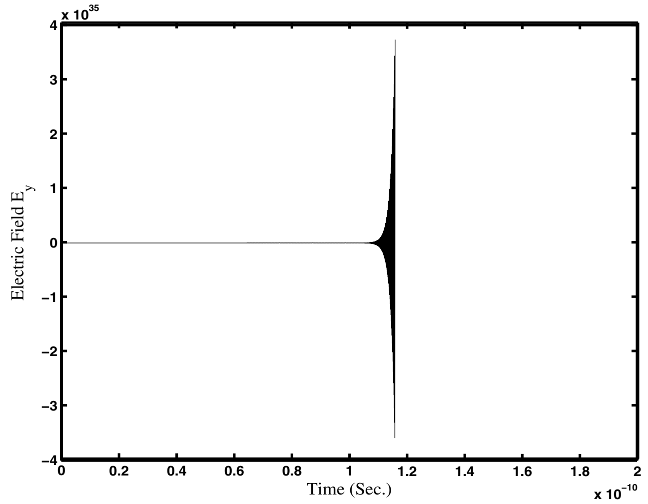
A. Numerical Verification of the Stability

First we investigate the stability of the proposed ADI-FETD method. Simulations were run for the homogeneous two-dimensional cavity with both the conventional and the proposed FETD having a time step that exceeds the limit determined by the stability condition for the conventional FETD, *i.e.* $\Delta t_{FETDMAX} = 5.5 \times 10^{-14} Sec.$ in our case. Fig. 2 shows the electric field recorded at the center of the cavity. $\Delta t = 5.6 \times 10^{-14} Sec.$ was used with the conventional FETD, while a 10 times larger time step $\Delta t = 56 \times 10^{-14} Sec.$ was used with the ADI-FETD scheme. As can be seen, the conventional FETD quickly becomes unstable [see Fig. 2(a)], while the ADI-FETD remains with stable solution [see Fig. 2(b)]. We also extended the simulation time to a much long period with the proposed scheme. No instability was observed.

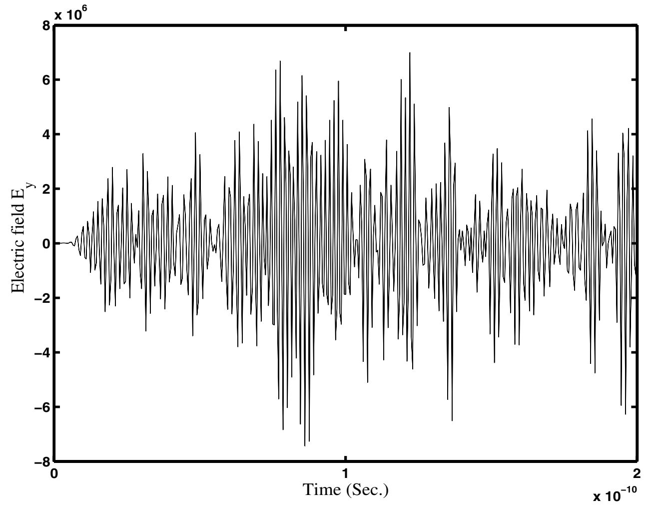
B. Numerical Accuracy Versus Time Step

Since the proposed ADI-FETD is proven to be stable for very large time steps towards an unconditionally stable scheme, the selection of the time step is no longer restricted by stability but by modeling accuracy. As a result, it is interesting and meaningful to investigate how the time step will affect accuracy.

For the comparision purpose, both the conventional FETD and ADI-FETD methods were used to simulate the cavity again. The time step $\Delta t_{FETD} = 5.4 \times 10^{-14} Sec.$ was chosen and fixed with the conventional FETD, while different values



(a)



(b)

Fig. 2. Time-domain electric fields at the center of the cavity recorded with the conventional FETD and the proposed ADI-FETD. (a) Conventional FETD solution that becomes unstable with $\Delta t = 5.6 \times 10^{-14} Sec.$(b) Proposed ADI-FETD solution with $\Delta t = 56 \times 10^{-14} Sec.$

of time step Δt_i were used with the proposed FETD to check for the accuracy. Table I presents the simulation results for the dominant mode which is TE_{32} in the cavity. The dominant mode is determined according to the frequency components of the excitation and its position. As can be seen, the relative errors of the ADI-FETD increase with the time step. These errors are completely due to the modeling accuracy of the numerical algorithm, such as the numerical dispersion. The tradeoff to the increased errors is, however, the reduction in the number of the iterations and the CPU time. By increasing the time step the conventional FETD

TABLE I

PROPOSED ADI-FETD SIMULATION RESULTS WITH DIFFERENT Δt

Analytical Result (GHz)	The proposed ADI-FETD scheme			
	$\Delta t_1 = 6\Delta t_{FETD}$		$\Delta t_2 = 10\Delta t_{FETD}$	
	Result (GHz)	Relative error	Result (GHz)	Relative error
540.59	550.30	1.85%	558.10	3.33%

solutions diverge (become unstable), while the proposed FETD continues to produce stable results with increasing errors that may or may not be acceptable depending on the applications and users' specifications.

V. CONCLUSIONS

We introduce a finite element time-domain method based on the alternating-direction implicit scheme. Using the ADI formulation, it was shown that the method is an unconditionally stable scheme. Numerical simulation shows that this method is very efficient, and the results agree very well with that of the conventional FETD method which uses a leap-frog scheme for time discretization.

We have explained the ADI-FETD method for a two-dimensional TE wave. However, our explanation can also be applied to a two-dimensional TM wave and can be extended in a general sense to a full three-dimensional wave.

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