Alternating-Direction Implicit Formulation of the Finite-Element Time-Domain Method

Masoud Movahhedi, *Student Member, IEEE*, Abdolali Abdipour, *Senior Member, IEEE*, Alexandre Nentchev, Mehdi Dehghan, and Siegfried Selberherr, *Fellow, IEEE*

***Abstract*—In this paper, two implicit finite-element time-domain (FETD) solutions of the Maxwell equations are presented. The first time-dependent formulation employs a time-integration method based on the alternating-direction implicit (ADI) method. The ADI method is directly applied to the time-dependent Maxwell curl equations in order to obtain an unconditionally stable FETD approach, unlike the conventional FETD method, which is conditionally stable. A numerical formulation for the 3-D ADI-FETD method is presented. For stability analysis of the proposed method, the amplification matrix is derived. Investigation of the proposed method formulation shows that it does not generally lead to a tri-diagonal system of equations. Therefore, the Crank–Nicolson FETD method is introduced as another alternative in order to obtain an unconditionally stable method. Numerical results are presented to demonstrate the effectiveness of the proposed methods and are compared to those obtained using the conventional FETD method.**

***Index Terms*—Alternating-direction implicit (ADI) technique, Crank–Nicolson (CN) method, finite-element time-domain (FETD) method, instability, Maxwell's equations, unconditional stability.**

I. Introduction

OVER THE past few years, considerable attention has been devoted to time-domain numerical methods to solve Maxwell's equations for the analysis of transient problems. Due to their potential to generate wideband data and model nonlinear materials, numerical simulation schemes for simulating electromagnetic transients have grown increasingly popular in recent years. Several methods can be used to calculate the time-domain solution of electromagnetic problems. The well-known one is the finite-difference time-domain (FDTD) algorithm, introduced by Yee in 1966 [1]. The FDTD method discretizes the time-dependent Maxwell curl equations using central differences in time and space and a leap-frog explicit scheme for time integration. Its principal advantage is ease of implementation. However, this method suffers from the well-known staircase problem, and its removal requires much more effort in the sacrifice of computational resources. The finite-element time-domain (FETD) method combines the advantages of a time-domain technique and the versatility of its spatial discretization procedure [2]. In contrast, the FETD method can easily handle both complex geometry and inhomogeneous media, which cannot be achieved by the FDTD scheme.

Over the past few years, a variety of FETD methods have been proposed [2]–[18]. These schemes fall into two categories. One directly discretizes Maxwell's equations, which typically results in an explicit finite-difference-like leap-frog scheme. These approaches are conditionally stable [4]–[8]. The other discretizes the second-order vector wave equation, also known as the curl–curl equation, obtained by eliminating one of the field variables from Maxwell's equations [9]–[18]. These solvers can be formulated to be unconditionally stable [9]–[13] or conditionally stable [14]–[18]. In an unconditionally stable scheme, the time step is not constrained by a stability criterion. However, it is limited by the required numerical accuracy in implementing the time derivatives of the electromagnetic fields. Therefore, if the minimum cell size in the computational domain is required to be much smaller than the wavelength, these schemes can be more efficient in terms of computer resources such as CPU time.

In some simulations using the FETD method, it is preferred that the first-order Maxwell equations are directly considered and solved. For instance, implementation of the complex frequency shifted perfectly matched layer in open-region electromagnetic problems, which has better performance than the conventional perfectly matched layer, is easier and more efficient when directly applied to Maxwell's curl equations [19]. However, the unconditionally stable methods for the FETD solution of the second-order vector wave equation are usually used. In this paper, we introduce two unconditionally stable vector FETD methods based on the alternating-direction implicit (ADI) and Crank–Nicolson (CN) schemes to directly solving first-order Maxwell's equations. The ADI technique was first introduced to solve Maxwell's curl equations using the finite-difference method. This algorithm is called the alternating-direction implicit finite-difference time-domain (ADI-FDTD) method [20], [21]. We previously applied the ADI-FETD method for solving the 2-D TE wave [22]. In this paper, we extend this approach to the 3-D wave and introduce the 3-D ADI-FETD method. Moreover, another alternative for time discretization to obtain an unconditionally stable method for the FETD solution of the Maxwell equations based on the CN scheme is presented. It will be shown that the ADI method can be considered as a perturbation of the implicit CN formulation.

Manuscript received September 30, 2006; revised January 5, 2007 and March 3, 2007. This work was supported in part by the Iran Telecommunication Research Center.

M. Movahhedi and A. Abdipour are with the Department of Electrical Engineering, AmirKabir University of Technology, Tehran, Iran (e-mail: movahhedi@aut.ac.ir; abdipour@aut.ac.ir).

A. Nentchev and S. Selberherr are with the Institut für Mikroelektronik, Technische Universität Wien, A-1040 Vienna, Austria.

M. Dehghan is with the Department of Applied Mathematics, AmirKabir University of Technology, Tehran, Iran.

Digital Object Identifier 10.1109/TMTT.2007.897777

ADI and CN schemes involve solving a linear system at each time step. When the ADI scheme is applied to a standard FDTD approach, the matrices are well structured and it renders a linear system that is only semi-implicit and easy to solve. That is, it leads to a 1-D solution of a tri-diagonal system of equations that must be factorized at each time step. However, this is neither the case for the ADI scheme, nor for the CN scheme with a finite-element approach. They lead to a fully implicit system. Several lumping techniques have been proposed in order to obtain explicit schemes without solving a linear system at each time step [14], [23]–[25]. Moreover, a recently developed approach avoids lumping altogether by constructing a set of orthogonal vector basis functions that yield a diagonal mass matrix [26], [27]. A most recent explicit FETD method, which is fundamentally different from traditional explicit FETD formulations for solving Maxwell's equations, has been introduced [28]. This new explicit FETD is derived from a recently developed FETD decomposition algorithm [29] by extending domain decomposition to the element level. With the element-level decomposition, no global system matrix has to be assembled and solved as required in the implicit FETD, and each element is related to its neighboring elements in an explicit manner.

Here, we explain the details of numerical formulations of the ADI- and CN-FETD solutions of the 3-D Maxwell equations. Moreover, some numerical results are provided to validate the proposed methods. This paper is organized in the following manner. In Section II, the essential principle of the ADI scheme for time discretization of time-dependent partial differential equations is presented. Section III describes the formulations of the proposed 3-D ADI-FETD method. Section IV presents and investigates the stability condition of the conventional and proposed schemes. The CN-FETD method as a more accurate unconditionally stable method is presented in Section V. In Section VI, the numerical results are shown. Finally, conclusions are presented in Section VII.

II. ADI PRINCIPLE

The ADI technique is well reported in the study of parabolic equations with finite elements [30]–[32]. In this paper, we use this technique to solve Maxwell's curl equations and the contribution is relevant to wave propagation (hyperbolic equations). The ADI technique takes its name from breaking up a single implicit time step into two half time steps. In the first half time step, an implicit evaluation is applied to one dimension and an explicit evaluation is applied to the other, assuming two dimensions in the problem statement. For the second half time step, the implicit and explicit evaluations are alternated, or switched, between the two dimensions. The dimensions to alternate between are typically spatial; however, temporal variables can also be used [33].

For explanation of the ADI method as a technique for the development of an implicit integration scheme, the time-dependent curl vector equations of Maxwell's equations are considered

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \frac{\mathbf{B}}{\mu} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}. \tag{1}$$

These equations can be cast into six scalar partial differential equations in Cartesian coordinates. We consider the following scalar equation from the above given system:

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right). \tag{2}$$

By applying the ADI principle, which is widely used in solving parabolic equations [34], the computation of (2) for the FETD solution marching from the nth time step to the $(n+1)$th time step is broken up into two computational sub-advancements: the advancement from the nth time step to the $(n+1/2)$th time step and the advancement from the $(n+1/2)$th time step to the $(n+1)$th time step. More specifically, the two substeps are as follows.

1) For the first half time step, i.e., at the $(n+1/2)$th time step, the first partial derivative on the right-hand side of (2), i.e., $\partial E_x/\partial y$, is replaced with its unknown pivotal values at the $(n+1/2)$th time step; while the second partial derivatives on the right-hand side, i.e., $\partial E_y/\partial x$, is replaced with its known values at the previous nth time step. In other words,

$$\frac{H_z^{n+\frac{1}{2}} - H_z^n}{\Delta t/2} = \frac{1}{\mu}\left(\frac{\partial E_x^{n+\frac{1}{2}}}{\partial y} - \frac{\partial E_y^n}{\partial x}\right). \tag{3}$$

2) For the second half time step, i.e., at the $(n+1)$th time step, the second term on the right-hand side, i.e., $\partial E_y/\partial x$, is replaced with its unknown pivotal values at the $(n+1)$th time step; while the first term, i.e., $\partial E_x/\partial y$, is replaced with its known values at the previous $(n+1/2)$th time step. In other words,

$$\frac{H_z^{n+1} - H_z^{n+\frac{1}{2}}}{\Delta t/2} = \frac{1}{\mu}\left(\frac{\partial E_x^{n+\frac{1}{2}}}{\partial y} - \frac{\partial E_y^{n+1}}{\partial x}\right). \tag{4}$$

The above two substeps represent the alternations in the FETD recursive computation directions in the sequence of the terms, i.e., the first and second terms. They result in the implicit formulations, as the right-hand side's of the equations contain the field values unknown and to be updated. The technique is then termed "the alternating direction implicit" technique. Attention should also be paid to the fact that no time-step difference (or lagging) between electric and magnetic field components is present in the formulations.

Applying the same procedure to all of the other five scalar differential equations of Maxwell's equations, one obtains the complete set of the implicit formula.

III. FORMULATIONS OF THE 3-D ADI-FETD SCHEME

The ADI-FETD solution of Maxwell's equations for analyzing full 3-D electromagnetic problems is described here. The Maxwell curl equations governing the solution of a 3-D problem in a lossless medium have been given by (1). In these equations, $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ is the electric field and $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$ is the magnetic flux density.

According to the ADI procedure for the time discretization, the following equations are obtained.

- $(n+(1/2))$th time step

$$\frac{1}{\mu}\left(\frac{B_x^{n+\frac{1}{2}}-B_x^n}{\Delta t/2}\right)=-\frac{1}{\mu}\left(\frac{\partial E_z^{n+\frac{1}{2}}}{\partial y}-\frac{\partial E_y^n}{\partial z}\right)$$
$$\frac{1}{\mu}\left(\frac{B_y^{n+\frac{1}{2}}-B_y^n}{\Delta t/2}\right)=-\frac{1}{\mu}\left(\frac{\partial E_x^{n+\frac{1}{2}}}{\partial z}-\frac{\partial E_z^n}{\partial x}\right)$$
$$\frac{1}{\mu}\left(\frac{B_z^{n+\frac{1}{2}}-B_z^n}{\Delta t/2}\right)=-\frac{1}{\mu}\left(\frac{\partial E_y^{n+\frac{1}{2}}}{\partial x}-\frac{\partial E_x^n}{\partial y}\right)$$
$$\epsilon\left(\frac{E_x^{n+\frac{1}{2}}-E_x^n}{\Delta t/2}\right)=\frac{1}{\mu}\left(\frac{\partial B_z^{n+\frac{1}{2}}}{\partial y}-\frac{\partial B_y^n}{\partial z}\right)-J_x^{n+\frac{1}{2}}$$
$$\epsilon\left(\frac{E_y^{n+\frac{1}{2}}-E_y^n}{\Delta t/2}\right)=\frac{1}{\mu}\left(\frac{\partial B_x^{n+\frac{1}{2}}}{\partial z}-\frac{\partial B_z^n}{\partial x}\right)-J_y^{n+\frac{1}{2}}$$
$$\epsilon\left(\frac{E_z^{n+\frac{1}{2}}-E_z^n}{\Delta t/2}\right)=\frac{1}{\mu}\left(\frac{\partial B_y^{n+\frac{1}{2}}}{\partial x}-\frac{\partial B_x^n}{\partial y}\right)-J_z^{n+\frac{1}{2}}. \quad (5)$$

- $(n+1)$th time step

$$\frac{1}{\mu}\left(\frac{B_x^{n+1}-B_x^{n+\frac{1}{2}}}{\Delta t/2}\right)=-\frac{1}{\mu}\left(\frac{\partial E_z^{n+\frac{1}{2}}}{\partial y}-\frac{\partial E_y^{n+1}}{\partial z}\right)$$
$$\frac{1}{\mu}\left(\frac{B_y^{n+1}-B_y^{n+\frac{1}{2}}}{\Delta t/2}\right)=-\frac{1}{\mu}\left(\frac{\partial E_x^{n+\frac{1}{2}}}{\partial z}-\frac{\partial E_z^{n+1}}{\partial x}\right)$$
$$\frac{1}{\mu}\left(\frac{B_z^{n+1}-B_z^{n+\frac{1}{2}}}{\Delta t/2}\right)=-\frac{1}{\mu}\left(\frac{\partial E_y^{n+\frac{1}{2}}}{\partial x}-\frac{\partial E_x^{n+1}}{\partial y}\right)$$
$$\epsilon\left(\frac{E_x^{n+1}-E_x^{n+\frac{1}{2}}}{\Delta t/2}\right)=\frac{1}{\mu}\left(\frac{\partial B_z^{n+\frac{1}{2}}}{\partial y}-\frac{\partial B_y^{n+1}}{\partial z}\right)-J_x^{n+1}$$
$$\epsilon\left(\frac{E_y^{n+1}-E_y^{n+\frac{1}{2}}}{\Delta t/2}\right)=\frac{1}{\mu}\left(\frac{\partial B_x^{n+\frac{1}{2}}}{\partial z}-\frac{\partial B_z^{n+1}}{\partial x}\right)-J_y^{n+1}$$
$$\epsilon\left(\frac{E_z^{n+1}-E_z^{n+\frac{1}{2}}}{\Delta t/2}\right)=\frac{1}{\mu}\left(\frac{\partial B_y^{n+\frac{1}{2}}}{\partial x}-\frac{\partial B_x^{n+1}}{\partial y}\right)-J_z^{n+1}. \quad (6)$$

Now we consider the finite-element solution of the above equations. The examined 3-D domain Ω in the xyz-volume is assumed to be discretized by a finite-element mesh composed of N_t tetrahedral elements, N_e edges, and N_f faces. In each point $\mathbf{r}$ of the element, Ω^e, the electric field $\mathbf{E}$, and the magnetic flux density $\mathbf{B}$ are approximated by edge and facet elements, respectively, as

$$\mathbf{E}^e(\mathbf{r},t)=\sum_{j=1}^{6}e_j(t)\mathbf{W}_j(\mathbf{r})$$
$$\mathbf{B}^e(\mathbf{r},t)=\sum_{j=1}^{4}b_j(t)\mathbf{F}_j(\mathbf{r}) \quad (7)$$

where $e_j(t)$ is the electric field circulation along the jth edge, $b_j(t)$ is the flux of the magnetic flux density through the jth face, $\mathbf{W}_j$ is the Whitney one-form vector basis function associated to the jth edge, and $\mathbf{F}_j$ is the Whitney two-form vector basis function associated to the jth face [35] such that

$$\mathbf{W}\in\mathcal{H}(\mathrm{curl};\Omega)=\left\{\mathbf{u}:\nabla\times\mathbf{u}\in\left[\mathcal{L}^2(\Omega)\right]^3\right\}$$
$$\mathbf{F}\in\mathcal{H}(\mathrm{div};\Omega)=\left\{\mathbf{u}:\nabla\cdot\mathbf{u}\in\mathcal{L}^2(\Omega)\right\}. \quad (8)$$

The Lebesgue spaces are defined by $\mathcal{L}^p(\Omega)=\{f:\|f\|_{\mathcal{L}^p}<\infty\}$ where

$$\|f\|_{\mathcal{L}^p}=\left(\int_\Omega|f|^p d\Omega\right)^{1/p}. \quad (9)$$

The space $\mathcal{L}^2(\Omega)$ is the space of all functions on domain Ω that are square integrable, which is often referred to as the space of functions with finite energy [36]. For vector functions, $\mathbf{f}:R^3\to R^3$, the corresponding space is denoted $[\mathcal{L}^2(\Omega)]^3$.

For Whitney one-forms, the basis functions are well known by now. For example, for the edge $\mathrm{ed}\{mn\}$, where m and n are nodes of the edge, it is

$$\mathbf{W}=\xi_m\nabla\xi_n-\xi_n\nabla\xi_m \quad (10)$$

where ξ_m is the Lagrange interpolation polynomial at vertex m [35]. Similarly, the vector basis functions for Whitney two-forms associated with a particular facet $fc\{mnp\}$, where m, n, and p are nodes of the face, can be written as

$$\mathbf{F}=2(\xi_m\nabla\xi_n\times\nabla\xi_p+\xi_n\nabla\xi_p\times\nabla\xi_m+\xi_p\nabla\xi_m\times\nabla\xi_n). \quad (11)$$

The Galerkin method is applied to the Maxwell curl equations (5) and (6) using the field approximations (7). Testing the first three scalar equations of (5) and (6) with basis function $\mathbf{F}_i$ and the second three with basis function $\mathbf{W}_i$ yield the following equations in the matrix form.

- $(n+(1/2))$th time step

$$[G^e]\frac{\{b\}^{n+\frac{1}{2}}-\{b\}^n}{\Delta t/2}=-\left([K_1^e]\{e\}^{n+\frac{1}{2}}+[K_2^e]\{e\}^n\right)$$
$$[C^e]\frac{\{e\}^{n+\frac{1}{2}}-\{e\}^n}{\Delta t/2}=[L_1^e]\{b\}^{n+\frac{1}{2}}+[L_2^e]\{b\}^n-\{q\}^{n+\frac{1}{2}}. \quad (12)$$

- $(n+1)$th time step

$$[G^e]\frac{\{b\}^{n+1}-\{b\}^{n+\frac{1}{2}}}{\Delta t/2}=-\left([K_1^e]\{e\}^{n+\frac{1}{2}}+[K_2^e]\{e\}^{n+1}\right)$$
$$[C^e]\frac{\{e\}^{n+1}-\{e\}^{n+\frac{1}{2}}}{\Delta t/2}=[L_1^e]\{b\}^{n+\frac{1}{2}}+[L_2^e]\{b\}^{n+1}-\{q\}^{n+1} \quad (13)$$

where the matrix entries are given by

$$
\begin{aligned}
G^e_{ij} &= \langle \mathbf{F}_i, \mu^{-1}\mathbf{F}_j\rangle_{\Omega^e} \\
&= \iiint\limits_{\Omega^e} \mu^{-1}\mathbf{F}_i\cdot\mathbf{F}_j d\Omega \\
K^e_{1ij} &= \langle \mathbf{F}_i, \mathbf{A}\rangle_{\Omega^e} \\
K^e_{2ij} &= \langle \mathbf{F}_i, \mathbf{B}\rangle_{\Omega^e} \\
\mathbf{W}_j &= W_{x_j}\hat{\mathbf{x}} + W_{y_j}\hat{\mathbf{y}} + W_{z_j}\hat{\mathbf{z}} \\
\mathbf{A} &= A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}} \\
A_x &= \mu^{-1}\left(\nabla\times\left(W_{z_j}\hat{\mathbf{z}}\right)\right)\cdot\hat{\mathbf{x}} \\
A_y &= \mu^{-1}\left(\nabla\times\left(W_{x_j}\hat{\mathbf{x}}\right)\right)\cdot\hat{\mathbf{y}} \\
A_z &= \mu^{-1}\left(\nabla\times\left(W_{y_j}\hat{\mathbf{y}}\right)\right)\cdot\hat{\mathbf{z}} \\
\mathbf{B} &= B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}} \\
B_x &= \mu^{-1}\left(\nabla\times\left(W_{y_j}\hat{\mathbf{y}}\right)\right)\cdot\hat{\mathbf{x}} \\
B_y &= \mu^{-1}\left(\nabla\times\left(W_{z_j}\hat{\mathbf{z}}\right)\right)\cdot\hat{\mathbf{y}} \\
B_z &= \mu^{-1}\left(\nabla\times\left(W_{x_j}\hat{\mathbf{x}}\right)\right)\cdot\hat{\mathbf{z}} \\
C^e_{ij} &= \langle \mathbf{W}_i, \epsilon\mathbf{W}_j\rangle_{\Omega^e} \\
L^e_{1ij} &= K^{eT}_{1ij} \\
L^e_{2ij} &= K^{eT}_{2ij} \\
q_i &= \langle \mathbf{W}_i, \mathbf{J}\rangle_{\Omega^e}.
\end{aligned} \tag{14}
$$

For tetrahedral elements, it can be easily seen that $\mathbf{A} = \mathbf{B} = 0.5(\nabla\times\mathbf{W}_j)$ so $[K^e_1] = [K^e_2]$ and $[L^e_1] = [L^e_2]$.

Equations (12) and (13) can be further simplified for efficient computation. By substituting the expressions for $\{e\}^{n+(1/2)}$ and $\{e\}^{n+1}$ presented by the second equation of (12) and (13) into their first equations and transferring the local equations to a global system, one obtains the following.

- $(n+(1/2))$th time step

$$
\begin{aligned}
&\left([G] + \frac{\Delta t^2}{4}[K_1][C]^{-1}[L_1]\right)\{b\}^{n+\frac{1}{2}} \\
&\quad= \left([G] - \frac{\Delta t^2}{4}[K_1][C]^{-1}[L_2]\right)\{b\}^{n} \\
&\qquad - \frac{\Delta t}{2}\left([K_1]+[K_2]\right)\{e\}^n + \frac{\Delta t^2}{4}[K_1][C]^{-1}\{q\}^{n+\frac{1}{2}} \\
&[C]\{e\}^{n+\frac{1}{2}} \\
&\quad= [C]\{e\}^n + \frac{\Delta t}{2}[L_1]\{b\}^{n+\frac{1}{2}} + \frac{\Delta t}{2}[L_2]\{b\}^n \\
&\qquad - \frac{\Delta t}{2}\{q\}^{n+\frac{1}{2}}.
\end{aligned} \tag{15}
$$

By solving the above equations, we first obtain the values of the magnetic flux density at the $(n+(1/2))$th time step. Thereafter, the values of the electric field can be directly calculated using the values of $\{b\}^{n+(1/2)}$. For the second half time step, we have the following.

- $(n+1)$th time step

$$
\begin{aligned}
&\left([G] + \frac{\Delta t^2}{4}[K_2][C]^{-1}[L_1]\right)\{b\}^{n+1} \\
&\quad= \left([G] - \frac{\Delta t^2}{4}[K_2][C]^{-1}[L_2]\right)\{b\}^{n+\frac{1}{2}} \\
&\qquad - \frac{\Delta t}{2}\left([K_1]+[K_2]\right)\{e\}^{n+\frac{1}{2}} + \frac{\Delta t^2}{4}[K_2][C]^{-1}\{q\}^{n+1} \\
&[C]\{e\}^{n+1} \\
&\quad= [C]\{e\}^{n+\frac{1}{2}} + \frac{\Delta t}{2}[L_1]\{b\}^{n+\frac{1}{2}} + \frac{\Delta t}{2}[L_2]\{b\}^{n+1} \\
&\qquad - \frac{\Delta t}{2}\{q\}^{n+1}.
\end{aligned} \tag{16}
$$

In this step, as in the previous one, the values of the magnetic flux density are first computed and then the values of the electric field are computed.

IV. Stability Analysis

In the linear theory of grid-based methods for the numerical solution of ordinary and partial differential equations, the success of numerical schemes is summed up in the well-known Lax equivalence theorem [37]: "consistency and stability imply convergence." For general FETD methods, the consistency is self-evident and assured by their formulations [2]. Subsequently, the question of stability is of paramount importance. Here, we present and investigate the stability condition of both conventional and proposed schemes for time discretization of the finite-element method.

One of the factors that affect the performance of a computational method is numerical dispersion. The proposed finite-element method, like other grid-based methods for solving Maxwell's equations, such as the FDTD method [38] and ADI-FDTD method [39], [40], exhibits numerical dispersion and numerical anisotropy due to the finite grid and finite time sampling. The numerical dispersion relation for the time-domain vector finite-element method has been derived on a 3-D hexahedral grid [36]. Investigation of the numerical dispersion of the ADI finite-element method will be considered as future work.

A. Conventional FETD Scheme

In the conventional method for time discretization of the Maxwell equations, time is discretized such that the electric degrees of freedom will be known at whole time steps and the magnetic degrees of freedom will be known at the half time steps. This is often refereed to as leap-frog method. Using this method for time discretization and following the analysis of [41] gives the equations

$$
\begin{aligned}
[G]\{b\}^{n+\frac{1}{2}} &= -\Delta t[K]\{e\}^n + [G]\{b\}^{n-\frac{1}{2}} \\
[C]\{e\}^{n+1} &= \Delta t[L]\{b\}^{n+\frac{1}{2}} + [C]\{e\}^n
\end{aligned} \tag{17}
$$

where $[K] = [K_1] + [K_2]$ and $[L] = [L_1] + [L_2] = [K]^T$. The matrices $[G]$ and $[C]$ are symmetric positive definite. The source term can be neglected for the stability analysis. These equations can be expressed in matrix form as

$$
\{x\}^{n+1} = [Q]\cdot\{x\}^n \tag{18}
$$

where

$$
\begin{aligned}
\{x\}^{n+1} &= \begin{Bmatrix} \{e\}^{n+1} \\ \{b\}^{n+\frac{1}{2}} \end{Bmatrix} \\
\{x\}^{n} &= \begin{Bmatrix} \{e\}^{n} \\ \{b\}^{n-\frac{1}{2}} \end{Bmatrix} \\
[Q] &= \begin{bmatrix} [I]-\Delta t^2[C]^{-1}[K]^T[G]^{-1}[K] & \Delta t[C]^{-1}[K]^T \\ -\Delta t[G]^{-1}[K] & [I] \end{bmatrix}.
\end{aligned} \tag{19}
$$

The matrix $[Q]$ in (18) is called the amplification matrix of the method. Stability of the above equation requires $\rho([Q]) \leq 1$, where $\rho([Q])$ is the spectral radius of $[Q]$. A tedious, but straightforward calculation shows that the eigenvalues of $[Q]$ are given by

$$\lambda = \frac{\tilde{\lambda} \pm i\sqrt{4-\tilde{\lambda}^2}}{2} \tag{20}$$

where $\tilde{\lambda} = 2 - \Delta t^2 \xi$ and ξ is an eigenvalue of the matrix $[C]^{-1}[K]^T[G]^{-1}[K]$. Equivalently, ξ and $\{x\}$ satisfy the generalized eigenvalue problem

$$[C]\{x\} = \xi [K]^T[G]^{-1}[K]\{x\}. \tag{21}$$

The matrix $[C]$ is symmetric positive definite and the matrix $[K]^T[G]^{-1}[K]$ is symmetric positive semidefinite. Thus, $\xi \geq 0$ and the eigenvalues λ of the amplification matrix $[Q]$ will have unit magnitude if and only if [36]

$$\Delta t \leq \frac{2}{\sqrt{\max(\xi)}}. \tag{22}$$

A similar bound on the time step for stability of the nonorthogonal grid finite-difference time-domain (NFDTD) schemes and the generalized Yee (GY) methods was derived in [41]. In these methods, structure of the amplification matrix is similar to the structure of the matrix $[Q]$ for the conventional FETD scheme.

B. Proposed ADI-FETD Scheme

For investigation of the stability condition of the proposed FETD method, we first derive the amplification matrix of the method. In general, (15) and (16) can be summarized as the following matrix form:

$$\{x\}^{n+\frac{1}{2}} = [P_1]\{x\}^n \tag{23}$$

(for advancement from the nth to $(n+(1/2))$th time step)

$$\{x\}^{n+1} = [P_2]\{x\}^{n+\frac{1}{2}} \tag{24}$$

(for advancement from $(n+(1/2))$th to $(n+1)$th time step).

The matrices $[P_1]$ and $[P_2]$ have been defined as (25), shown at the bottom of this page.

It is easy to show that $[K_1] = [K_2] = (1/2)[K]$ and $[L_1] = [L_2] = (1/2)[K]^T$. Thus, $[P_1] = [P_2] = [P]$.

Combination of the above two equations reads

$$\{x\}^{n+1} = [P_2][P_1]\{x\}^n \tag{26}$$

or simply

$$\{x\}^{n+1} = [P]^2\{x\}^n. \tag{27}$$

By checking the magnitude of the eigenvalues of $[P]^2$, one can determine whether the proposed scheme is unconditionally stable; if the magnitudes of all the eigenvalues of $[P]^2$ are equal to or less than unity, the proposed scheme is unconditionally stable; otherwise it is potentially unstable [30].

Direct finding of eigenvalues of the amplification matrix appears to be very difficult. Therefore, an indirect approach can be used with which the ranges of the eigenvalues can be determined. For instance, the Schur–Cohn–Fujiwa criterion can be applied, where the characteristic polynomial of $[P]^2$, with its roots being the eigenvalues, is examined [42]. This investigation and analytical proof for unconditional stability can be considered as an open problem and will remain a topic for future research. It is important to note that numerical results obtained from many simulations show that the scheme is stable even for large time steps. Thus, from the implementation aspect, the method can be considered as an unconditionally stable scheme.

V. Alternative Description of the ADI Method

One of the principal advantages of the ADI-FDTD method is that it renders a system that is only semi-implicit. That is, it leads to a 1-D solution of a tri-diagonal system of equations that must be factorized at each time step. Since only the 1-D problem is being solved, the additional computational cost is significantly reduced. As a result, there is a tendency to sacrifice accuracy of the ADI time-integration to maintain a semi-implicit solution procedure for the FDTD method [43]–[47]. Generally, in the presented method (the ADI-FETD method), this process is lost

$$[P_1] = \begin{bmatrix} [\mathrm{Coeff}_1]^{-1}\left([C] - \frac{\Delta t^2}{4}[L_1][G]^{-1}[K_2]\right) & [\mathrm{Coeff}_1]^{-1}\left(\frac{\Delta t}{2}[L_1] + \frac{\Delta t}{2}[L_2]\right) \\ -[\mathrm{Coeff}_2]^{-1}\left(\frac{\Delta t}{2}[K_1] + \frac{\Delta t}{2}[K_2]\right) & [\mathrm{Coeff}_2]^{-1}\left([G] - \frac{\Delta t^2}{4}[K_1][C]^{-1}[L_2]\right) \end{bmatrix}$$
$$[P_2] = \begin{bmatrix} [\mathrm{Coeff}_3]^{-1}\left([C] - \frac{\Delta t^2}{4}[L_2][G]^{-1}[K_2]\right) & [\mathrm{Coeff}_3]^{-1}\left(\frac{\Delta t}{2}[L_1] + \frac{\Delta t}{2}[L_2]\right) \\ -[\mathrm{Coeff}_4]^{-1}\left(\frac{\Delta t}{2}[K_1] + \frac{\Delta t}{2}[K_2]\right) & [\mathrm{Coeff}_4]^{-1}\left([G] - \frac{\Delta t^2}{4}[K_2][C]^{-1}[L_2]\right) \end{bmatrix}$$
$$[\mathrm{Coeff}_1] = [C] + \frac{\Delta t^2}{4}[L_1][G]^{-1}[K_1], \quad [\mathrm{Coeff}_2] = [G] + \frac{\Delta t^2}{4}[K_1][C]^{-1}[L_1]$$
$$[\mathrm{Coeff}_3] = [C] + \frac{\Delta t^2}{4}[L_2][G]^{-1}[K_1], \quad [\mathrm{Coeff}_4] = [G] + \frac{\Delta t^2}{4}[K_2][C]^{-1}[L_1] \tag{25}$$

since, for an arbitrary tessellation, the field coefficients cannot be decoupled into Cartesian projections. Consequently, a fully implicit procedure is resulted.

However, this is not the case with the special case of an orthogonal Cartesian mesh (orthogonal hexahedral element) for the finite-element method. The use of mass lumping techniques in this situation leads to a system with well-structured matrices [36] and, hence, the resulting linear systems can be readily solved by using methods such as ADI-FDTD. In fact, the use of mass lumping techniques for solving the proposed time-domain finite-element method with the orthogonal hexahedral element results in a semi-implicit procedure. Moreover, the proposed method can be more efficient than the other possible unconditionally stable schemes in some applications such as hybrid methods. Hybrid methods, which combine the desirable features of two or more different techniques, are being developed to analyze complex electromagnetic problems that cannot be otherwise resolved conveniently and/or accurately by using the methods individually. One of the most efficient hybrid methods is the finite-element finite-difference time-domain (FE-FDTD) hybrid method. In this method, the FDTD is used to treat large regions with less complexities, while using the FETD method is used to handle complex boundaries and structures [48]–[50]. An alternative to this hybrid method is the combination of unconditionally stable FDTD and FETD methods. The ADI-FDTD method is always used as an unconditionally stable method for the finite-difference method. By using the finite-element method whose time discretization is derived from the same scheme as in the FDTD method (i.e., the proposed method in this paper), the overall performance of the unconditionally stable hybrid method is improve. Therefore, the proposed FETD method can be efficiently used in the unconditionally stable FE-FDTD hybrid method. Another application where the ADI scheme can be efficiently used is the hybrid finite-element boundary-integral (FE-BI) method. This hybrid method is a powerful numerical technique for solving open-region electromagnetic problems [51], [52]. This method uses an artificial boundary to divide the infinite solution domain into interior and exterior regions in which fields are represented using finite elements and boundary integrals, respectively. The resulting hybrid method permits accurate and efficient analysis of complex electromagnetic phenomena, especially those involving inhomogeneous media. In [53] and [54], it has been shown that the ADI method is an efficient method for solving problems with nonlocal boundary conditions (integral boundary conditions). Therefore, the proposed method, which is based on the ADI method, can also be efficiently implemented for the FE-BI method.

Recent analysis shows that although the ADI scheme for time discretization has second-order accuracy in time, the method exhibits a splitting error associated with the square of the time-step size [45]. Having an asymptotic second-order accuracy, the splitting error becomes dominant in regions with larger spatial derivatives. This can be detrimental for modeling problems where strong near-field coupling occurs and/or structures contain field singularities such as in tips and corners. It can be shown that the ADI method applied to the Maxwell equations is a perturbation of the implicit CN formulation [43]. The CN scheme is a well-known implicit algorithm in numerical computation. The CN algorithm advances time by a full time-step size with one marching procedure.

3-D Maxwell's equations can be written as

$$\frac{\partial\{u\}}{\partial t} = [A]\{u\} + [B]\{u\} \tag{28}$$

where $\{u\} = [E_x, E_y, E_z, H_x, H_y, H_z]^T$, $[A]$, and $[B]$ are partial derivative operators defined in [47].

Applying the CN scheme at time step $n + (1/2)$, we get

$$\begin{aligned}\frac{\{u\}^{n+1} - \{u\}^n}{\Delta t} &= \frac{1}{2}\left([A]\{u\}^{n+1} + [A]\{u\}^n\right)\\ &\quad + \frac{1}{2}\left([B]\{u\}^{n+1} + [B]\{u\}^n\right) + O(\Delta t)^2.\end{aligned} \tag{29}$$

The above equation can be factorized as

$$\begin{aligned}&\left([I] - \frac{\Delta t}{2}[A] - \frac{\Delta t}{2}[B]\right)\{u\}^{n+1}\\ &\quad = \left([I] + \frac{\Delta t}{2}[A] + \frac{\Delta t}{2}[B]\right)\{u\}^n + O(\Delta t)^3\end{aligned} \tag{30}$$

or

$$\begin{aligned}&\left([I] - \frac{\Delta t}{2}[A]\right)\left([I] - \frac{\Delta t}{2}[B]\right)\{u\}^{n+1}\\ &\underbrace{= \left([I] + \frac{\Delta t}{2}[A]\right)\left([I] + \frac{\Delta t}{2}[B]\right)\{u\}^n}_{\text{One-step approximation}}\\ &\underbrace{+ \frac{\Delta t^2}{4}[A][B]\left(\{u\}^{n+1} - \{u\}^n\right) + O(\Delta t)^3}_{\text{Error}=O(\Delta t)^3}.\end{aligned} \tag{31}$$

The "One-step approximation" can be written in two stages, which become equivalent to the two stages of the ADI scheme [47]. Since the ADI method solves "One-step approximation" instead of (31) (CN method), it introduces a splitting error of the form

$$\frac{\Delta t^2}{4}[A][B]\left(\{u\}^{n+1} - \{u\}^n\right) \tag{32}$$

to the solution. The effect of this splitting error depends on three factors, i.e.: 1) the time-step size (Δt^2 factor); 2) the spatial derivatives of the field ($[A][B]$ factor); and 3) the temporal variation of the field ($(\{u\}^{n+1} - \{u\}^n)$ factor). When field variation and/or the time step size is large, the splitting error becomes more pronounced.

To obtain the CN-FETD method formulation, the Maxwell curl equations is projected into its weak form using the Galerkin method and spatially discretizing using finite elements. Based on the CN formulation for time discretization, the following equations are derived:

$$\begin{aligned}[G]\frac{\{b\}^{n+1} - \{b\}^n}{\Delta t} &= -[K]\frac{\{e\}^{n+1} + \{e\}^n}{2}\\ [C]\frac{\{e\}^{n+1} - \{e\}^n}{\Delta t} &= [L]\frac{\{b\}^{n+1} + \{b\}^n}{2}\end{aligned} \tag{33}$$

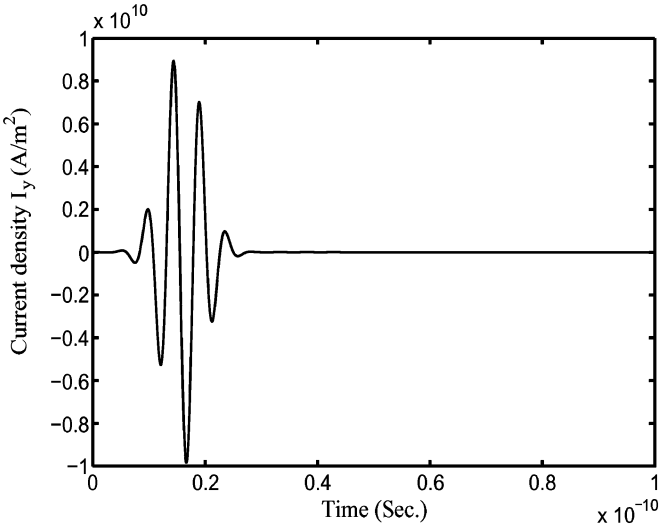


Fig. 1. Time variations of current density, I_y used as an excitation for the 3-D rectangular cavity.

where $K_{ij} = \langle \mathbf{F}_i, \mu^{-1}\nabla \times \mathbf{W}_j \rangle_\Omega$ and $[L] = [K]^T$. Equation (33) can be further simplified as

$$\left([G] + \frac{\Delta t^2}{4}[K][C]^{-1}[L]\right)\{b\}^{n+1} = \left([G] - \frac{\Delta t^2}{4}[K][C]^{-1}[L]\right)\{b\}^{n} - \frac{\Delta t}{2}[K]\{e\}^n$$

$$[C]\{e\}^{n+1} = [C]\{e\}^n + \frac{\Delta t}{2}[L]\{b\}^{n+1} + \frac{\Delta t}{2}[L]\{b\}^n. \quad (34)$$

As can be seen from (34), the CN formulation is very similar to the ADI scheme for the FETD solution of Maxwell's equations. From Section IV, in the ADI formulation [see (15) and (16)], we have $[K_1] = [K_2] = (1/2)[K]$ and $[L_1] = [L_2] = (1/2)[L]$.

VI. Numerical Results

Here, to validate the accuracy and stability of the proposed 3-D ADI- and CN-FETD methods, a simple example of the excitation of a lossless 3-D rectangular cavity with perfectly conducting walls is studied. The dimensions of the cavity are 1.0 mm $\times$ 0.5 mm $\times$ 1.5 mm. For comparative purposes, this cavity is simulated and its resonant frequencies are obtained with both the conventional and proposed FETD methods. In the conventional FETD method, a leap-frog scheme is applied for time discretization. The computation was carried out using a mesh with tetrahedral elements. This mesh contains 27550 tetrahedra and 51710 edges, which is used for the ADI- and CN-FETD methods and also for the conventional FETD method. On the cavity surfaces, which are assumed to be perfectly conducting walls since the tangential components of the electric field is zero, we impose a homogeneous Dirichlet boundary condition, i.e., we set to zero all the electric field circulations associated with edges belonging to these surfaces.

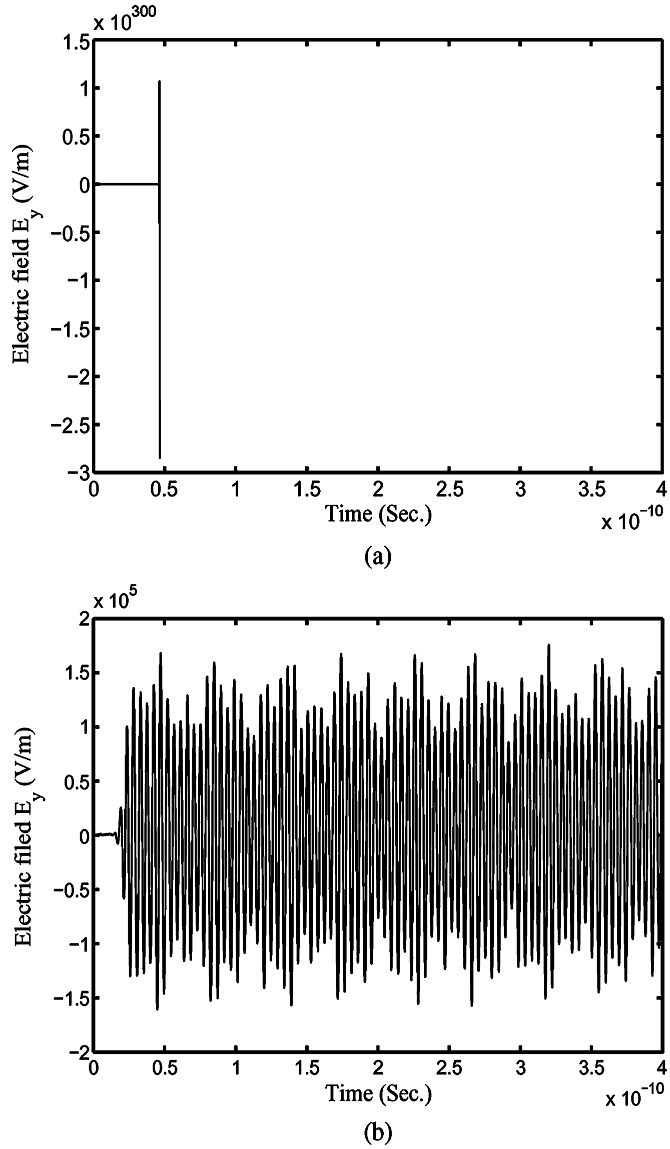


Fig. 2. Time-domain electric fields at the center of the cavity recorded with the conventional FETD method and the proposed ADI-FETD method. (a) Conventional FETD solution that becomes unstable with $\Delta t = 1.05 \times \Delta t_{\mathrm{FETDMAX}}$. (b) Proposed 3-D ADI-FETD solution with $\Delta t = 15 \times \Delta t_{\mathrm{FETDMAX}}$.

For each half time step, the updating of the electric and magnetic fields requires solving two matrix equations. The system will be written generically as $[A]\{x\} = \{b\}$. Since this matrix is not time dependent, it can be factorized once before the step-by-step procedure to obtain an explicit scheme. For matrix factorization, we use the direct solver in the SGI's scientific computing software library (SCSL), which turns out to be a highly efficient direct solver. Note that the factorization is performed only once and that only forward and backward substitutions are needed in each time step. Noting this on (15) and (16) shows that for forming the general matrix system $[A]\{x\} = \{b\}$ to update the magnetic field, the matrix $[C]^{-1}$ must be computed. We also use the SCSL routines, which calculate the inverse after LU factorization, to compute the matrix inverse, directly.

The resonant frequencies can be obtained by launching a time signal and applying the Fourier transform on the time response.

TABLE I
PROPOSED ADI-FETD, CN-FETD, AND CONVENTIONAL FETD SIMULATION RESULTS WITH DIFFERENT TIME STEPS

Analytical solution (GHz)	Δt	ADI-FETD		CN-FETD		FETD	
		Resonant frequency (GHz)	Relative error (%)	Resonant frequency (GHz)	Relative error (%)	Resonant frequency (GHz)	Relative error (%)
180.20	Δt_{FETD}	181.32	0.62	179.30	-0.49	179.01	-0.66
	$4\Delta t_{FETD}$	182.01	1.00	181.70	0.83	N/A	N/A
	$8\Delta t_{FETD}$	184.06	2.14	183.5	1.83	N/A	N/A
	$12\Delta t_{FETD}$	185.41	2.89	184.9	2.60	N/A	N/A

An excitation sinusoidally modulated Gaussian is used as current density in this simulation given by

$$\mathbf{J} = A\cos(wt)\exp\left(-\left(\frac{t-t_0}{\tau}\right)^2\right)\hat{\mathbf{y}} \tag{35}$$

where $A = 1\times10^{10}, w = 2\times\pi\times(2.1\times10^{11}), t_0 = 16\times10^{-12}$, and $\tau = 5\times10^{-12}$. Fig. 1 shows the time variations of this current density.

A. Numerical Verification of the Stability

First, we investigate the stability of the proposed ADI- and CN-FETD methods. Simulations were run for the homogeneous 3-D rectangular cavity with both the conventional and proposed methods having a time step that exceeds the limit determined by the stability condition for the conventional FETD method, i.e., in our case, $\Delta t_{\text{FETDMAX}} \approx 4.1\times10^{-14}$ s. Fig. 2 shows the electric field recorded at the center of the cavity. $\Delta t = 1.05\times\Delta t_{\text{FETDMAX}}$ was used with the conventional FETD method, while a 15 times larger time step $\Delta t = 15\times\Delta t_{\text{FETDMAX}}$ was used with the ADI-FETD scheme. As can be seen, the conventional FETD quickly becomes unstable [see Fig. 2(a)], while the ADI FETD remains with a stable solution [see Fig. 2(b)]. We also extended the simulation time to a much longer period with the proposed scheme. No instability was observed. For another scheme, the CN-FETD method, the same large time step was used and the stable solution was shown.

B. Numerical Accuracy Versus Time Step

Since the proposed ADI- and CN-FETD methods are shown to be stable for very large time steps, the selection of the time step is no longer restricted by stability, but by modeling accuracy. As result, it is interesting and meaningful to investigate how the time step will affect accuracy.

For comparative purposes, both the conventional FETD method and proposed methods were used to simulate the cavity again. The time step $\Delta t_{\text{FETD}} = 4.0\times10^{-14}$ s was chosen and fixed with the conventional FETD method, while different values of time step Δt_i were used with the proposed ADI- and CN-FETD methods to check for the accuracy. Table I presents the simulation results for the dominant mode, which is TE^z_{101} in the cavity. The dominant mode is determined according to the frequency components of the excitation and its position. As can be seen, the relative errors of the proposed unconditionally stable methods increase with the time step, while this increment is more for the ADI method in contrast with the CN scheme. These errors are completely due to the modeling accuracy of the numerical algorithm such as the numerical dispersion. The tradeoff to the increased errors is, however, the reduction in the number of iterations and CPU time. By increasing the time step, the conventional FETD solutions become unstable, while the proposed FETD method continues to produce stable results.

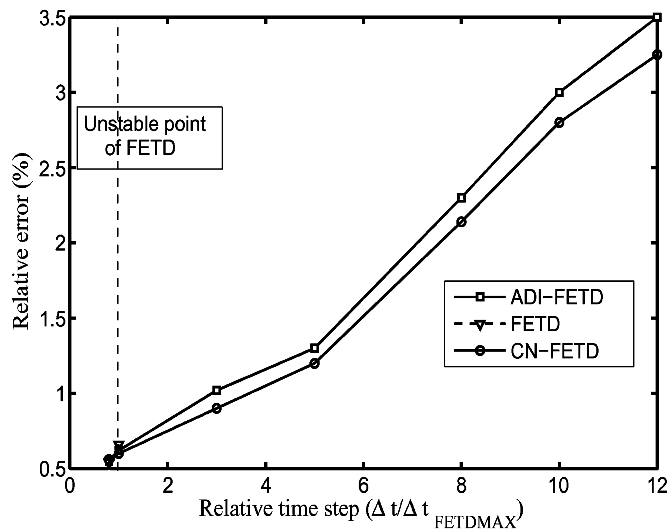


Fig. 3. Relative errors of the conventional FETD method, proposed ADI-FETD, and CN-FETD methods as a function of relative time step $\Delta t/\Delta t_{\text{FETDMAX}}$. The dashed line represents the unstable point of the conventional FETD scheme.

Fig. 3 illustrates a plot of the errors for another excited mode of the cavity TE^z_{102} versus the discrete time step Δt computed using the conventional FETD method and the proposed ADI- and CN-FETD methods. For clarity, a relative time-step $\Delta t/\Delta t_{\text{FETDMAX}}$ is used. As can be seen, at low $\Delta t/\Delta t_{\text{FETDMAX}}$, the errors of both the conventional FETD method and the proposed FETD methods are almost the same. However, after $\Delta t/\Delta t_{\text{FETDMAX}} = 1.0$, the conventional FETD solutions diverge, while the proposed FETD methods continue to produce stable results with increasing errors that may or may not be acceptable depending on the applications and users' specifications.

VII. CONCLUSION

This paper has introduced the 3-D ADI-FETD method for solving first-order Maxwell's equations. Using an ADI scheme, which is applied directly to the Maxwell curl equations for time discretization, leads to an implicit FETD method. For stability analysis, the amplification matrix of the proposed method was derived, and it was shown, with numerical simulations, that an unconditionally stable scheme is achievable. Moreover, numerical simulation shows that this method is very efficient and the results agree very well with those of the conventional FETD method, which uses a leap-frog scheme for time discretization.

Investigation of the ADI-FETD formulation shows that this method, unlike the ADI-FDTD method, does not generally lead to a tri-diagonal system of equations. Hence, we used a CN time integration and introduced unconditionally stable CN-FETD method. When field variation and/or the time step size is large, the CN-FETD method becomes more accurate than the ADI-FETD method.

Since in some electromagnetic simulations it is preferred that the first-order Maxwell equations are directly solved, the proposed unconditionally stable methods can be very efficient and useful. They can decrease the simulation time because the time step is no longer restricted by the numerical stability, but by the modeling accuracy of the FETD algorithm.

Acknowledgment

The authors would like to thank the three reviewers of this paper for their constructive comments and helpful suggestions.

References

[1] K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas Propag.*, vol. AP-14, no. 8, pp. 302–307, Aug. 1966.

[2] J. F. Lee, R. Lee, and A. C. Cangellaris, "Time-domain finite element methods," *IEEE Trans. Antennas Propag.*, vol. 45, no. 3, pp. 430–442, Mar. 1997.

[3] K. Mahadevan, R. Mittra, and P. M. Vaidya, "Use of Whitney's edge and face elements for efficient finite element time domain solution of Maxwell's equations," *J. Electromagn. Waves Applicat.*, vol. 8, no. 9/10, pp. 1173–1191, 1994.

[4] K. Choi, S. J. Salon, K. A. Connor, L. F. Libelo, and S. Y. Hahn, "Time domain finite analysis of high power microwave aperture antennas," *IEEE Trans. Magn.*, vol. 31, no. 5, pp. 1622–1625, May 1995.

[5] M. Feliziani and F. Maradei, "An explicit-implicit solution scheme to analyze fast transients by finite elements," *IEEE Trans. Magn.*, vol. 33, no. 3, pp. 1452–1455, Mar. 1997.

[6] A. C. Cangellaris, C. C. Lin, and K. K. Mei, "Point-matched time-domain finite element methods for electromagnetic radiation and scattering," *IEEE Trans. Antennas Propag.*, vol. AP-35, no. 10, pp. 1160–1173, Oct. 1987.

[7] J. T. Elson, H. Sangani, and C. H. Chan, "An explicit time-domain method using three-dimensional Whitney elements," *Microw. Opt. Technol. Lett.*, vol. 7, pp. 607–610, Sep. 1994.

[8] M. F. Wong, O. Picon, and V. F. Hanna, "A finite-element method based on Whitney forms to solve Maxwell equations in the time-domain," *IEEE Trans. Magn.*, vol. 31, no. 5, pp. 1618–1621, May 1995.

[9] G. Mur, "The finite-element modeling of three-dimensional time-domain electromagnetic fields in strongly inhomogeneous media," *IEEE Trans. Magn.*, vol. 28, no. 3, pp. 1130–1133, Mar. 1992.

[10] J. F. Lee and Z. Sacks, "Whitney elements time domain (WETD) methods," *IEEE Trans. Magn.*, vol. 31, no. 5, pp. 1325–1329, May 1995.

[11] S. D. Gedney and U. Navsariwala, "An unconditionally stable finite-element time-domain solution of the vector wave equation," *IEEE Microw. Guided Wave Lett.*, vol. 5, no. 5, pp. 332–334, May 1995.

[12] W. P. Capers, Jr., L. Pichon, and A. Razek, "A 3-D finite element method for the modeling of bounded and unbounded electromagnetic problems in the time domain," *Int. J. Numer. Modeling*, vol. 13, pp. 527–540, 2000.

[13] R. Fernades and G. Fairweather, "An alternating direction Galerkin method for a class of second order hyperbolic equations in two space dimensions," *SIAM J. Numer. Anal.*, vol. 28, no. 5, pp. 1265–1281, 1991.

[14] D. R. Lynch and K. D. Paulsen, "Time-domain integration of the Maxwell equations on finite elements," *IEEE Trans. Antennas Propag.*, vol. 38, no. 12, pp. 1933–1942, Dec. 1990.

[15] J. F. Lee, "WETD—A finite-element time-domain approach for solving Maxwell's equations," *IEEE Microw. Guided Wave Lett.*, vol. 4, no. 1, pp. 11–13, Jan. 1994.

[16] Z. Sacks and J. F. Lee, "A finite-element time-domain method using prism elements for microwave cavities," *IEEE Trans. Electromagn. Compat.*, vol. 37, no. 4, pp. 519–527, Nov. 1995.

[17] J. M. Jin, M. Zunoubi, K. C. Donepudi, and W. C. Chew, "Frequency domain and time-domain finite-element solution of Maxwell's equations using spectral Lanczos decomposition method," *Comput. Methods Appl. Mech. Eng.*, vol. 169, pp. 279–296, 1999.

[18] D. A. White, "Orthogonal vector basis functions for time domain finite element solution of the vector wave equation," *IEEE Trans. Magn.*, vol. 35, no. 5, pp. 1458–1461, May 1999.

[19] M. Movahhedi, A. Abdipour, H. Ceric, A. Sheikholeslami, and S. Selberherr, "Optimization of the perfectly matched layer for the finite-element time-domain method," *IEEE Microw. Wireless Compon. Lett.*, vol. 17, no. 1, pp. 10–12, Jan. 2007.

[20] T. Namiki, "A new FDTD algorithm based on alternating-direction implicit method," *IEEE Trans. Microw. Theory Tech.*, vol. 47, no. 10, pp. 2003–2007, Oct. 1999.

[21] S. Gonzalez Garcia, T. W. Lee, and S. C. Hagness, "On the accuracy of the ADI-FDTD method," *IEEE Antennas Wireless Propag. Lett.*, vol. 1, no. 1, pp. 31–34, Dec. 2002.

[22] M. Movahhedi, A. Nentschev, H. Ceric, A, Abdipour, and S. Selberherr, "A finite element time-domain algorithm based on the alternating-direction implicit method," in *Proc. 36th Eur. Microw. Conf.*, Manchester, U.K., 2006, pp. 1–4.

[23] J. M. Jin, M. Zunoubi, K. C. Donepudi, and W. C. Chew, "Frequency-domain and time-domain finite-element solution of Maxwell's equations using spectral Lanczos decomposition method," *Comput. Methods Appl. Mech. Eng.*, vol. 169, pp. 279–296, 1999.

[24] A. Bossavit and L. Kettunen, "Yee-like schemes on a tetrahedral mesh," *In. J. Numer. Modeling*, vol. 12, pp. 129–142, 1999.

[25] G. Cohen*, Higher Order Numerical Methods for Transient Wave Equations*. Berlin, Germany: Springer-Verlag, 2002.

[26] D. A. White, "Orthogonal vector basis functions for time domain finite element solution of the vector wave equation," *IEEE Trans. Mag.*, vol. 35, no. 5, pp. 1458–1461, May 1999.

[27] D. Jiao and J. M. Jin, "Three-dimensional orthogonal vector basis functions for time-domain finite element solution of vector wave equations," *IEEE Trans. Antennas Propag.*, vol. 51, no. 1, pp. 59–66, Jan. 2003.

[28] Z. Lou and J. M. Jin, "A new explicit time-domain finite-element method based on element-level decomposition," *IEEE Trans. Antennas Propag.*, vol. 54, no. 10, pp. 2990–2999, Oct. 2006.

[29] Z. Lou and J. M. Jin, "A novel dual-field time-domain finite-element domain-decomposition method for computational electromagnetics," *IEEE Trans. Antennas Propag.*, vol. 54, no. 6, pp. 1850–1862, Jun. 2006.

[30] G. D. Smith*, Numerical Solution of Partial Differential Equations*. Oxford, U.K.: Oxford Univ. Press, 1978.

[31] W. Dai, Y. Zhang, and R. Nassar, "A hybrid finite element alternating direction implicit method for solving parabolic differential equations on multilayers with irregular geometry," *J. Comput. Appl. Math.*, vol. 117, pp. 1–16, 2000.

[32] D. Lindholm, "A finite element method for solution of the three dimensional time dependent heat conduction equation with application for heating of steels in reheating furnaces," *Numer. Heat Transfer*, vol. 25, pt. A, pp. 155–172, 1999.

[33] T. Namiki and K. Ito, "A new FDTD algorithm free from the CFL condition restraint for a 2D-TE wave," in *IEEE AP-S Symp. Dig.*, Orlando, FL, Jul. 1999, vol. 1, pp. 192–195.

[34] D. W. Peaceman and H. H. Rachford, "The numerical solution of parabolic and elliptic differential equations," *J. Soc. Ind. Appl. Math.*, vol. 42, no. 3, pp. 28–41, 1955.

[35] A. Bossavit, "Whitney forms: A class of finite elements for three-dimensional computations in electromagnetism," *Proc. Inst. Elect. Eng.*, vol. 135, pp. 493–500, Nov. 1988.

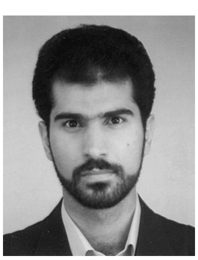
[36] D. A. White, "Discrete time vector finite element methods for solving Maxwell's equations on 3-D unstructured grids," Ph.D. dissertation, Dept. Appl. Sci., Univ. California at Davis, Davis, CA, 1997.

[37] G. F. Carey and J. T. Oden*, Finite Elements III: Computational Aspects*. Englewood Cliffs, NJ: Prentice-Hall, 1984.

[38] A. Taflove*, Computational Electrodynamics: The Finite-Difference Time-Domain Method*. Norwood, MA: Artech House, 1996.

[39] F. Zheng and Z. Chen, "Numerical dispersion analysis for the unconditionally stable 3-D ADI-FDTD method," *IEEE Trans. Microw. Theory Tech.*, vol. 49, no. 5, pp. 1006–1009, May 2001.

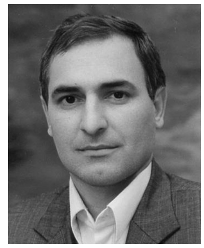
[40] F. Zheng and Z. Chen, "A study of the numerical dispersion relation for the 2-D ADI-FDTD method," *IEEE Microw. Wireless Compon. Lett.*, vol. 13, no. 9, pp. 405–407, Sep. 2003.
[41] S. D. Gedney and J. A. Roden, "Numerical stability of nonorthogonal FDTD method," *IEEE Trans. Antennas Propag.*, vol. 48, no. 2, pp. 231–239, Feb. 2000.
[42] E. I. Jury, *Inners and Stability of Dynamic Systems*. New York: Wiley, 1974.
[43] S. G. Garcia, T.-W. Lee, and S. C. Hagness, "On the accuracy of the ADI-FDTD method," *IEEE Antennas Wireless Propag. Lett.*, vol. 1, pp. 31–34, Dec. 2002.
[44] G. Sun and C. W. Trueman, "Approximate Crank–Nicolson schemes for the 2-D finite-difference time-domain method for TE_z waves," *IEEE Trans. Antennas Propag.*, vol. 52, no. 11, pp. 2963–2972, Nov. 2004.
[45] G. Sun and C. W. Trueman, "Unconditionally-stable FDTD method based on Crank–Nicolson scheme for solving three-dimensional Maxwell equations," *Electron. Lett.*, vol. 40, no. 10, pp. 300–301, May 2004.
[46] S. Wang, F. L. Teixeira, and J. Chen, "An iterative ADI-FDTD with reduced splitting error," *IEEE Microw. Wireless Compon. Lett.*, vol. 15, no. 2, pp. 92–94, Feb. 2005.
[47] J. Lee and B. Fornbergb, "Some unconditionally stable time stepping methods for the 3-D Maxwell's equations," *J. Comput. Appl. Math.*, vol. 166, pp. 479–523, 2004.
[48] D. Koh, H. Lee, and T. Itoh, "A hybrid full-wave analysis of via-hole grounds using finite-difference and finite-element time-domain methods," *IEEE Trans. Microw. Theory Tech.*, vol. 45, no. 12, pp. 2217–2223, Dec. 1997.
[49] T. Rylander and A. Bondeson, "Application of stable FEM-FDTD hybrid to scattering problems," *IEEE Trans. Antennas Propag.*, vol. 50, no. 2, pp. 141–144, Feb. 2002.
[50] N. V. Venkatarayalu, G. Y. Beng, and L.-W. Li, "On the numerical errors in the 2-D FE/FDTD algorithm for different hybridization schemes," *IEEE Microw. Wireless Compon. Lett.*, vol. 14, no. 4, pp. 168–170, Apr. 2004.
[51] D. Jiao, M. Lu, E. Michielssen, and J. M. Jin, "A fast time-domain finite-element-boundary integral method for electromagnetic analysis," *IEEE Trans. Antennas Propag.*, vol. 49, no. 10, pp. 1453–1461, Oct. 2001.
[52] D. Jiao, A. A. Ergin, B. Shanker, E. Michielssen, and J. M. Jin, "A fast higher-order time-domain finite element boundary integral method for 3-D electromagnetic scattering analysis," *IEEE Trans. Antennas Propag.*, vol. 50, no. 9, pp. 1192–1202, Sep. 2002.
[53] M. Dehghan, "Alternating direction implicit methods for two-dimensional diffusion with a non-local boundary condition," *Int. J. Comput. Math.*, vol. 27, no. 3, pp. 349–366, 1999.
[54] M. Dehghan, "A new ADI technique for two-dimensional parabolic equation with an integral condition," *Comput. Math. Appl.*, vol. 43, pp. 1477–1488, 2002.



Masoud Movahhedi (S'06) was born in Yazd, Iran, in 1976. He received the B.Sc. degree in electrical engineering from Sharif University of Technology, Tehran, Iran, in 1998, the M.Sc. degree in electrical engineering from AmirKabir University of Technology (Tehran Polytechnic), Tehran, Iran, in 2000, and is currently working toward the Ph.D. degree in electrical engineering at AmirKabir University of Technology.

From December 2005 to September 2006, he was a Visiting Student with the Institute for Microelectronics, Technische Universität Wien, Vienna, Austria. His research interests are computer-aided design of microwave integrated circuits, computational electromagnetic, semiconductor high-frequency RF modeling, and interconnect simulations.

Mr. Movahhedi is a reviewer for the IEEE MICROWAVE AND WIRELESS COMPONENTS LETTERS. He was the recipient of the GAAS-05 Fellowship presented by the GAAS Association to young graduate researchers for his paper presented at GAAS2005. He was also the recipient of the Electrical Engineering Department Outstanding Student Award in 2005 and 2006.



Abdolali Abdipour (M'97–SM'06) was born in Alashtar, Iran, in 1966. He received the B.Sc. degree in electrical engineering from Tehran University, Tehran, Iran, in 1989, the M.Sc. degree in electronics from Limoges University, Limoges, France, in 1992, and the Ph.D. degree in electronic engineering from Paris XI University, Paris, France, in 1996.

He is currently a Professor with the Electrical Engineering Department, AmirKabir University of Technology (Tehran Polytechnic), Tehran, Iran. He is also currently Head of the Electrical Engineering Department, AmirKabir University of Technology and Director of the Radio Communication Center of Excellence. He has authored or coauthored over 115 papers in refereed journals and local and international conferences. He authored *Noise in Electronic Communication: Modeling, Analysis and measurement* (AmirKabir Univ. Press, 2005, in Persian) and *Transmission Lines* (Nahre Danesh Press, 2006, in Persian). His research areas include wireless communication systems (RF technology and transceivers), RF/microwave/millimeter-wave circuit and system design, electromagnetic (EM) modeling of active devices and circuits, high-frequency electronics (signal and noise), nonlinear modeling, and analysis of microwave devices and circuits.

Alexandre Nentchev was born in Sofia, Bulgaria, in 1971. He received the Diplomingenieur degree in electrical engineering from the Technische Universität Wien, Vienna, Austria, in 2004, and is currently working on his Ph.D. degree at the Institut für Mikroelektronik, Technische Universität Wien, Vienna, Austria.

His scientific interests include 3-D interconnect simulation of multilevel wired very large scale integration (VLSI) circuits and software technology.

Mehdi Dehghan was born in Eghlid-Fars, Iran, in 1957.

He is currently an Associate Professor with the Department of Applied Mathematics, AmirKabir University of Technology, Tehran, Iran. His research interests include numerical solutions of partial differential (and integral) equations, numerical integration, numerical linear algebra, and difference equations. He has authored or coauthored approximately 100 papers in refereed journals and has presented several papers at local and international conferences.

Siegfried Selberherr (M'79–SM'84–F'93) was born in Klosterneuburg, Austria, in 1955. He received the Diplomingenieur degree in electrical engineering and Doctoral degree in technical sciences from the Technische Universität Wien, Vienna, Austria, in 1978 and 1981, respectively.

Since 1984, he has held the Venia Docendi on computer-aided design. Since 1988, he has been the Chair Professor of the Institut für Mikroelektronik, Technische Universität Wien. From 1998 to 2005, he was Dean of the Fakultät für Elektrotechnik und Informationstechnik, Technische Universität Wien. His current research interests are modeling and simulation of problems for microelectronics engineering.