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Letter

Gauge-invariant Wigner equation for electromagnetic fields: Strong and weak formulation

Clemens Etl *, Mauro Ballicchia, Mihail Nedjalkov, Hans Kosina

Institute for Microelectronics, TU Wien, Gußhausstraße 27-29/E360, Vienna, 1040, Austria

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ABSTRACT

Gauge-invariant Wigner theory describes the quantum-mechanical evolution of charged particles in phase space, which is spanned by position and kinetic momentum. This approach uses the electromagnetic field variables instead of the electrodynamic potentials. Several approaches to derive a gauge-invariant Wigner evolution equation have been reported, which are generally complex. This work presents a new formulation for a single electron in a general electromagnetic field, which simplifies existing formulations. First, a gauge-dependent equation is derived using Moyal's formulation. A transformation of the Wigner function introduced by Stratonovich yields the strong form of the gauge-invariant equation. Expressing the pseudo-differential operators by integral operators gives the weak form of the gauge-invariant equation. An analysis of the different properties of the gauge-invariant equation is given, as well as the different requirements for the regularity and asymptotic behavior of the strong and weak forms.

1. Introduction

The quantum evolution of a charged particle in an electromagnetic (EM) field [1–4], which is relevant to areas such as nanoelectronics [5–15], can be described using various formulations, including gauge-invariant quantum mechanics [16–18], Wigner theory [19], and phase space trajectories governed by Fredholm integral equations [20]. Such theoretical frameworks are also fundamental to the description of entanglement [21] and thus highly relevant for modern applications in quantum cryptography [22] and correlated systems involving Coulomb interactions [23]. The most common description of a quantum mechanical system is given by the Schrödigner equation, where the particle's state is represented by the wave function ψ. It is generalized for mixed states by the von Neumann equation for the density matrix ρ. Both approaches yield the probability density of the particle's location. To evaluate the probability density of the momentum, these functions can be transformed by the unitary Fourier transform involving momentum variables.

According to Heisenberg's uncertainty principle, it is impossible to determine a particle's location and momentum precisely at the same time. Consequently, a common probability density function in phase space does not exist. Nonetheless, the Wigner formalism assesses the density using both the location and momentum. The Wigner function is introduced by applying a Weyl transform to the density matrix ρ,

$$f_{\mathrm{w}}(\mathbf{p},\mathbf{r}) = C\int \mathrm{d}\mathbf{s}\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{p}\right)\rho\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2}\right), \tag{1}$$

where a normalization constant $C=(2\pi\hbar)^{-3}$ is introduced. As f_{w} is normalized but can assume negative values, it represents a quasi-probability density in phase space. The foundation for describing the evolution of f_{w} was laid by Wigner, who transferred the density matrix using a Weyl transform, which consists of a coordinate transform followed by a Fourier transform. However, Wigner's work did not include the equation that describes the evolution of a quantum mechanical system for a general potential V.

The first crucial step to find such an equation was done by Groenewold [24], who investigated the correspondence between observables and their quantum operators. Unlike classical physics, the ordering of the operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ is crucial as they do not commute. Thus, a convention on the ordering must be established. In the Wigner formalism, a symmetric ordering is used that weighs every arrangement of the operators equally. For this case, Groenewold found the correspondence

$$a(\mathbf{p},\mathbf{r})\sin\left[\frac{\hbar}{2}\left(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}-\overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{r}}\right)\right]b(\mathbf{p},\mathbf{r}) \leftrightarrow \frac{\mathrm{i}}{2}(\hat{a}\hat{b}-\hat{b}\hat{a}), \tag{2}$$

* Corresponding author.
E-mail address: etl@iue.tuwien.ac.at (C. Etl).

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for two observables a and b that depend on location and momentum. The sine function is used in the form of its series expansion, which represents a pseudo-differential operator that is now known as the Moyal bracket

$$\{\{a,b\}\} := \frac{2}{\hbar} a(\mathbf{p},\mathbf{r}) \sin\left[\frac{\hbar}{2}\left(\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\right)\right] b(\mathbf{p},\mathbf{r}), \tag{3}$$

which Moyal [25] applied to the Hamiltonian H and the Wigner function to derive the first formulation of the Wigner equation, which is now known as Moyal's equation:

$$\frac{\partial f_{\mathrm{w}}}{\partial t} = \{\{H, f_{\mathrm{w}}\}\}. \tag{4}$$

In the classical limit $\hbar \to 0$, the Moyal bracket reduces to the Poisson bracket, which establishes the connection to classical mechanics.

For EM fields, Hamiltonian formulations of quantum mechanics depend on the set of chosen scalar and vector potentials $(\mathbf{A}, \phi)$ and are thus gauge-dependent: The freedom of changing the gauge affects the mathematical formulation and the numerical treatment of the given problem. This also holds for the standard Wigner quantum mechanics, derived in a phase space spanned by position and canonical momentum. Furthermore, the canonical momentum $\mathbf{p}$ is also gauge-dependent, meaning it cannot be measured, but only serves as an auxiliary variable, which undermines the connection between the theory and its experimental verifiability. This can be avoided by replacing $\mathbf{p}$ with the kinetic momentum $\mathbf{P} = \mathbf{p} - q\mathbf{A}(\mathbf{r})$, as it is an observable and therefore uniquely defined for a given problem. Few approaches have been made to derive a formulation of a gauge-invariant Wigner equation (GIWE) [16,17,26–29].

One way to modify f_{w} to make it gauge-invariant by using the vector potential was given by Stratonovich [18,30,31]. For this purpose, a gauge-invariant density matrix $\overline{\rho}$ is introduced,

$$\overline{\rho}(\mathbf{s},\mathbf{r}) = \exp\left[-\frac{\mathrm{i}q}{2\hbar}\int_{-1}^{1} \mathbf{A}\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right) \cdot \mathbf{s}\, \mathrm{d}\tau\right] \tilde{\rho}(\mathbf{s},\mathbf{r}), \tag{5}$$

where

$$\tilde{\rho}(\mathbf{s},\mathbf{r}) = \rho\left(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}\right). \tag{6}$$

The Weyl–Stratonovich (WS) function F_{s} is defined as the Fourier transform of $\overline{\rho}$ with respect to the variable $\mathbf{s}$,

$$F_{\mathrm{s}}(\mathbf{P},\mathbf{r}) := C \int \mathrm{d}\mathbf{s} \exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s} \cdot \mathbf{P}\right) \overline{\rho}(\mathbf{s},\mathbf{r}). \tag{7}$$

Despite that the Stratonovich function was suggested seven decades ago, there is little experience with this formalism and its relationships with the standard theory. In general, both are obtained from the density matrix and the corresponding von-Neumann evolution equation, applying the Weyl or the Weyl–Stratonovich transform, respectively. Here, we show that Moyal's approach conveniently gives rise to a complete picture of the internal interdependencies of these theories: The Moyal sine function of phase space differential operators provides the standard strong formulation of the gauge-dependent Wigner equation.

In [31], the GIWE was derived directly from the von Neumann equation, whereas in this work we use Moyal's formulation of the WE, like in the historical evolution of the standard Wigner theory, described around (1)–(4). This leads to a strong formulation in which the interaction with the EM field is represented by differential operators. We will prove the equivalence of the weak and strong formulations and investigate the specific requirements of the physical quantities involved.

In the following section, we will derive a gauge-dependent formulation of the WE (GDWE) using Moyal's approach. In Section 3, the Weyl–Stratonovich transform is introduced, where we prove its gauge-invariance and show the properties needed for deriving the GIWE. In Section 4, the GIWE is derived from the GDWE. Both the strong and the weak formulation of the GIWE are given. The results and their physical implications are summarized in Section 5.

2. Gauge-dependent Wigner equations

The Hamiltonian for a charged particle with mass m and charge q in a general EM field is given by

$$H(\mathbf{p},\mathbf{r}) = \frac{1}{2m}[\mathbf{p} - q\mathbf{A}(\mathbf{r})]^2 + q\phi(\mathbf{r}). \tag{8}$$

The EM field is represented by the electrodynamic potentials $\mathbf{A}$ and ϕ as

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{9}$$

The terms in (8) that do not depend on the momentum $\mathbf{p}$ are summarized in the modified potential

$$\tilde{V}(\mathbf{r}) := q\phi(\mathbf{r}) + \frac{q^2}{2m}\mathbf{A}^2(\mathbf{r}), \tag{10}$$

so that H can be written as

$$H(\mathbf{p},\mathbf{r}) = \frac{\mathbf{p}^2}{2m} - \frac{q}{m}\mathbf{p} \cdot \mathbf{A}(\mathbf{r}) + \tilde{V}(\mathbf{r}). \tag{11}$$

We will insert the Hamiltonian (11) into (4) to derive the strong formulation of the GIWE. Eq. (4) can be reformulated by applying the angle subtraction formula for the sine.

$$\sin\left[\frac{\hbar}{2}\left(\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\right)\right] = \sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}}\right)\cos\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\right) - \cos\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}}\right)\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\right) \tag{12}$$

The lowest terms of the series expansions on sine and cosine are given as

$$\cos\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\right) = 1 - \frac{\hbar^2}{8}\left(\overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\right)^2 + \dots, \tag{13}$$

$$\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\right) = \frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}} - \frac{\hbar^3}{48}\left(\overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\right)^3 + \dots. \tag{14}$$

As $\tilde{V}$ is independent of $\mathbf{p}$, the application of $\overleftarrow{\nabla}_{\mathbf{p}}$ in (13) and (14) yields zero. Thus, only the 1 in the cosine gives a non-zero contribution, so that we can write

$$\tilde{V}\sin\left[\frac{\hbar}{2}(\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}})\right] = \tilde{V}\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}}\right). \tag{15}$$

Since $H(\mathbf{p},\mathbf{r})$ is a quadratic polynomial in $\mathbf{p}$, only the first summand of both the sine and cosine expansion yields a non-zero result. For the linear term, we obtain

$$\mathbf{p} \cdot \mathbf{A}\sin\left[\frac{\hbar}{2}(\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}})\right] = \mathbf{p} \cdot \mathbf{A}\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}}\right) - \frac{\hbar}{2}\mathbf{A} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\cos\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}}\right), \tag{16}$$

and for the quadratic term

$$\mathbf{p}^2\sin\left[\frac{\hbar}{2}(\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{r}})\right] = \mathbf{p}^2\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}}\right) - \hbar\mathbf{p} \cdot \overrightarrow{\nabla}_{\mathbf{r}}\cos\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}}\right). \tag{17}$$

Inserting (11) into (4) and using (15)–(17) yields the gauge-dependent WE (GDWE):

Theorem 1 (Strong formulation of the GDWE). *Let $\tilde{\rho}_0$ be the density matrix at $t = 0$ in an EM field with smooth electrodynamic potentials $(\mathbf{A}, \phi)$. For the Wigner function with the initial condition*

$$f_{\mathrm{w}}^0(\mathbf{p},\mathbf{r}) = C\int \mathrm{d}\mathbf{s}\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s} \cdot \mathbf{p}\right)\tilde{\rho}_0(\mathbf{s},\mathbf{r}), \tag{18}$$

the evolution of f_{w} is given by

$$\frac{\partial f_{\mathrm{w}}}{\partial t} = \left[\frac{1}{\hbar m}(\mathbf{p} - q\mathbf{A})^2 + \frac{2q}{\hbar}\phi\right]\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}}\right) f_{\mathrm{w}} - \frac{1}{m}(\mathbf{p} - q\mathbf{A}) \cdot \cos\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{p}}\right)\nabla_{\mathbf{r}} f_{\mathrm{w}}. \tag{19}$$

Interestingly, the kinetic momentum $\mathbf{p} - q\mathbf{A}$ appears twice in this equation. The second term on the right-hand side resembles a diffusion term. The potentials $\mathbf{A}$ and ϕ must be smooth in the given form.

To achieve gauge-invariance for the WE, a simple variable transform $\mathbf{p} \mapsto \mathbf{P} + q\mathbf{A}(\mathbf{r})$ in (19) does not suffice, because the additional term $q\mathbf{A}(\mathbf{r})$ does not compensate for the change of the density matrix under gauge transforms. Stratonovich's approach [30] to solving this issue was a modification of the Weyl transform. In the next section, we will explore this transformation and show its necessity for deriving the GIWE.

3. Weyl–Stratonovich transform

A gauge transform for given potentials $(\mathbf{A}, \phi)$ can be expressed using a scalar function χ, so that the new potentials

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \quad \phi' = \phi - \frac{\partial \chi}{\partial t} \tag{20}$$

represent the same EM field:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{21}$$

The density matrix $\tilde{\rho}'$, which depends on the gauge, can be evaluated using $\tilde{\rho}$ of another gauge as

$$\tilde{\rho}'(\mathbf{s}, \mathbf{r}, t) = \exp\left\{-\frac{1}{\mathrm{i}\hbar} q \left[\chi\left(\mathbf{r} + \frac{\mathbf{s}}{2}, t\right) - \chi\left(\mathbf{r} - \frac{\mathbf{s}}{2}, t\right)\right]\right\} \tilde{\rho}(\mathbf{s}, \mathbf{r}, t). \tag{22}$$

This means that when the Wigner function is expressed in terms of the kinetic momentum $\mathbf{P}$, i.e., $F_{\mathrm{w}}(\mathbf{P}, \mathbf{r}, t) := f_{\mathrm{w}}(\mathbf{P} + q\mathbf{A}, \mathbf{r}, t)$, it remains gauge-dependent, see Appendix A.1. To remove the gauge-dependence, we use Stratonovich's approach, which we introduce in the following transformation.

Definition 1 (Weyl–Stratonovich (WS) transform)**.** Let $\mathbf{A}$ be a vector potential and $f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. The WS transform is defined as

$$\mathcal{S} : f \mapsto C \int \mathrm{d}\mathbf{s} \exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s} \cdot \left[\mathbf{P} + \frac{q}{2}\int_{-1}^{1} \mathrm{d}\tau \mathbf{A}\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right)\right]\right\} f(\mathbf{s}, \mathbf{r}). \tag{23}$$

The Stratonovich function is defined as the WS transform of the density matrix $\tilde{\rho}(\mathbf{s}, \mathbf{r})$, i.e.,

$$F_{\mathrm{s}}(\mathbf{P}, \mathbf{r}) := C \int \mathrm{d}\mathbf{s} \exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s} \cdot \left[\mathbf{P} + \frac{q}{2}\int_{-1}^{1} \mathrm{d}\tau \mathbf{A}\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right)\right]\right\} \tilde{\rho}(\mathbf{s}, \mathbf{r}). \tag{24}$$

For F_{s} we can show

Lemma 3.2. *Let $\mathbf{A}$ be a vector potential and ρ a solution of the von Neumann equation. Then, the Stratonovich function F_{s} is invariant under gauge transformations.*

For the proof, check Appendix A.2. When we replace all f_{w} by F_{s} in (19), we can eliminate the potentials $(\mathbf{A}, \phi)$ and obtain an equation that contains only the fields $\mathbf{E}$ and $\mathbf{B}$. For this purpose, f_{w} is transformed back to the density matrix using an inverse Weyl transform

$$\mathcal{W}^{-1} : f_{\mathrm{w}}(\mathbf{p}, \mathbf{r}) \mapsto \int \mathrm{d}\mathbf{p} \exp\left(-\frac{1}{\mathrm{i}\hbar}\mathbf{s} \cdot \mathbf{p}\right) f_{\mathrm{w}}(\mathbf{p}, \mathbf{r}) \quad \left(= \tilde{\rho}(\mathbf{s}, \mathbf{r})\right), \tag{25}$$

followed by a WS transform. This concatenation yields a transformation $\mathcal{T} := \mathcal{S} \circ \mathcal{W}^{-1}$ that satisfies

$$\mathcal{T}(f_{\mathrm{w}}) = F_{\mathrm{s}}. \tag{26}$$

We will apply this transformation to both sides of (19), but first, we need to investigate its properties in more detail.

The linearity of $\mathcal{T}$ follows directly from the linearity of the Fourier transform

$$\mathcal{T}(a f_{\mathrm{w}} + b g_{\mathrm{w}}) = a F_{\mathrm{s}} + b G_{\mathrm{s}}, \tag{27}$$

where a and b can be functions of $\mathbf{r}$ and t. Using the sinc function $\mathrm{sinc}(x) = \sin(x)/x$, the transformation of the differential operators is given by

$$\mathcal{T}\left(\frac{\partial f_{\mathrm{w}}}{\partial t}\right) = \frac{\partial F_{\mathrm{s}}}{\partial t} - q\frac{\partial \mathbf{A}}{\partial t} \cdot \mathrm{sinc}\left(\frac{\hbar}{2}\overline{\nabla}_{\mathbf{r}} \cdot \overline{\nabla}_{\mathbf{P}}\right)\nabla_{\mathbf{P}} F_{\mathrm{s}}, \tag{28}$$

$$\mathcal{T}(\nabla_{\mathbf{r}} f_{\mathrm{w}}) = \nabla_{\mathbf{r}} F_{\mathrm{s}} - q\nabla_{\mathbf{r}}(\overline{\nabla}_{\mathbf{P}} \cdot \mathbf{A})\,\mathrm{sinc}\left(\frac{\hbar}{2}\overline{\nabla}_{\mathbf{r}} \cdot \overline{\nabla}_{\mathbf{P}}\right) F_{\mathrm{s}}, \tag{29}$$

$$\mathcal{T}(\nabla_{\mathbf{p}} f_{\mathrm{w}}) = \nabla_{\mathbf{P}} F_{\mathrm{s}}, \tag{30}$$

which is shown in Appendix A.3. Eq. (28) shows an important difference between f_{w} and F_{s} with respect to stationary states. While the condition $\partial f_{\mathrm{w}}/\partial t = 0$ has different solutions for different choices of $\mathbf{A}$, the solutions of $\partial F_{\mathrm{s}}/\partial t = 0$ remain the same. Only when $\mathbf{A}$ is time-independent are the stationary states for both functions the same.

The term $\mathbf{p} f_{\mathrm{w}}$ transforms as

$$\mathcal{T}(\mathbf{p} f_{\mathrm{w}}) = \left[\mathbf{P} + \frac{q\hbar}{2}\nabla_{\mathbf{r}}\left(\overline{\nabla}_{\mathbf{P}} \cdot \mathbf{A}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overline{\nabla}_{\mathbf{r}} \cdot \overline{\nabla}_{\mathbf{P}}\right) + q\mathbf{A}\,\mathrm{sinc}\left(\frac{\hbar}{2}\overline{\nabla}_{\mathbf{r}} \cdot \overline{\nabla}_{\mathbf{P}}\right)\right] F_{\mathrm{s}}, \tag{31}$$

see Appendix A.4.

We apply $\mathcal{T}$ to both sides of the GDWE (19) and use the properties (27)–(31) to derive the strong formulation of the GIWE.

4. Gauge-invariant Wigner equations

Since the derivation is elaborate and contains many terms, we discuss only the steps in this section. The detailed derivation of each step is given in Appendix B.1.

4.1. Strong formulation

1. The linearity of $\mathcal{T}$ enables us to transform each term in (19) one by one. We use (28)–(31) to transform all f_{w}'s into F_{s}'. In the remaining steps, all terms containing $\mathbf{A}$ and ϕ are replaced by terms containing $\mathbf{E}$ and $\mathbf{B}$.
2. We derive the identity
$$\begin{aligned}\mathbf{a} \times \mathbf{B} &= \mathbf{a} \times (\nabla_{\mathbf{r}} \times \mathbf{A}) \\ &= \nabla_{\mathbf{r}}(\mathbf{a} \cdot \mathbf{A}) - (\mathbf{a} \cdot \nabla_{\mathbf{r}})\mathbf{A},\end{aligned} \tag{32}$$
which we will use later, where $\mathbf{a}$ stands for $\mathbf{P}$ or $\overline{\nabla}_{\mathbf{P}}$ respectively.
3. One term contains $\mathbf{A}^2$, which we have to split up to be able to make use of the identity (32). We use the addition formula to rewrite it as
$$\begin{aligned}\mathbf{A}^2 \sin\left(\frac{\hbar}{2}\overline{\nabla}_{\mathbf{r}} \cdot \overline{\nabla}_{\mathbf{P}}\right) &= \mathbf{A}' \cdot \mathbf{A}'' \sin\left[\frac{\hbar}{2}\left(\overline{\nabla}_{\mathbf{r}}{}' + \overline{\nabla}_{\mathbf{r}}{}''\right) \cdot \overline{\nabla}_{\mathbf{P}}\right] \\ &= 2\left[\mathbf{A}\sin\left(\frac{\hbar}{2}\overline{\nabla}_{\mathbf{r}} \cdot \overline{\nabla}_{\mathbf{P}}\right)\right] \cdot \left[\mathbf{A}\cos\left(\frac{\hbar}{2}\overline{\nabla}_{\mathbf{r}} \cdot \overline{\nabla}_{\mathbf{P}}\right)\right],\end{aligned} \tag{33}$$
where the product rule was considered by marking $\mathbf{A}$ and $\overline{\nabla}_{\mathbf{r}}$ with $'$ and $''$ to identify which ∇-operator acts on which vector potential.
4. We also want all of the pseudo-differential operators to be either of type sinc or sinc$'$. Therefore, we use
$$\sin(x) = x\,\mathrm{sinc}(x), \tag{34}$$
$$\begin{aligned}\cos(x) &= \sin'(x) \\ &= [x\,\mathrm{sinc}(x)]' \\ &= \mathrm{sinc}(x) + x\,\mathrm{sinc}'(x).\end{aligned} \tag{35}$$
to replace the sines and cosines, where we set $x = \hbar/2\overline{\nabla}_{\mathbf{r}} \cdot \overline{\nabla}_{\mathbf{P}}$.
5. Steps 1 to 4 yield many terms, where some of them cancel each other. The term containing $\partial\mathbf{A}/\partial t$, which arises through the transformation of $\partial f_{\mathrm{w}}/\partial t$, can be combined with $\nabla_{\mathbf{r}}\phi$ so that we can use $\mathbf{E} = -\nabla_{\mathbf{r}}\phi - \partial\mathbf{A}/\partial t$. For the remaining terms, we use the identity (32). In this way, we can indeed eliminate all potentials.
6. Simplifying the remaining terms and using the definition of the Lorentz force $\mathbf{F} = q\mathbf{E} + qm^{-1}\mathbf{P} \times \mathbf{B}$ yields

Theorem 2 (Strong formulation of the GIWE)**.** *Let $\tilde{\rho}_0$ be the density matrix at $t = 0$ in an EM field $(\mathbf{E}, \mathbf{B})$. For the Stratonovich function with the initial condition*

$$F_{\mathrm{s}}^0(\mathbf{P}, \mathbf{r}) = C \int \mathrm{d}\mathbf{s} \exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s} \cdot \left[\mathbf{P} + \frac{q}{2}\int_{-1}^{1} \mathrm{d}\tau \mathbf{A}\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right)\right]\right\} \tilde{\rho}_0(\mathbf{s}, \mathbf{r}), \tag{36}$$

the evolution of F_s is given by

$$\frac{\partial F_s}{\partial t} + \frac{\mathbf{P}}{m} \cdot \nabla_{\mathbf{r}} F_s + \mathbf{F} \cdot \operatorname{sinc}\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right) \nabla_{\mathbf{P}} F_s$$
$$= -\frac{q\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}} \times \mathbf{B}\right) \operatorname{sinc}'\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right)$$
$$\cdot \left[\overrightarrow{\nabla}_{\mathbf{r}} - q\left(\overrightarrow{\nabla}_{\mathbf{P}} \times \mathbf{B}\right) \operatorname{sinc}\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right)\right] F_s. \tag{37}$$

In the classical limit $\hbar \to 0$, the right-hand side vanishes, and it holds $\operatorname{sinc}(0) = 1$, which gives the classical Liouville equation

$$\frac{\partial F_s}{\partial t} + \frac{\mathbf{P}}{m} \cdot \nabla_{\mathbf{r}} F_s + \mathbf{F} \cdot \nabla_{\mathbf{P}} F_s = 0. \tag{38}$$

The term containing $\nabla_{\mathbf{r}} F_s$ can be interpreted as a generalized diffusion term:

$$\left[\frac{\mathbf{P}}{m} + \frac{q\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}} \times \mathbf{B}\right) \operatorname{sinc}'\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right)\right] \cdot \nabla_{\mathbf{r}} F_s. \tag{39}$$

In (B.6), one can see that the transformation of the diffusion term of f_w introduces two terms that do not contain $\nabla_{\mathbf{r}} F_s$. This happens due to the transformation (29) of $\nabla_{\mathbf{r}}$, where a second term is introduced that contains $\nabla_{\mathbf{r}}(\overrightarrow{\nabla}_{\mathbf{P}} \cdot \mathbf{A})$. I.e., the diffusion term of f_w is gauge-dependent and only equal to the diffusion term of F_s when $\mathbf{A}$ is spatially constant, which is equivalent to having no magnetic field. This enables the evaluation of the particle diffusion, which is not provided for the gauge-dependent formulation.

The right-hand side of (37) consists of a linear term in $\mathbf{B}$ and a quadratic term in $\mathbf{B}$. As far as magnetism is concerned, the linear term describes the paramagnetic effect and the quadratic term the diamagnetic effect. Depending on the material under consideration, the quadratic term can be either negligible or dominant. When modeling light-matter interaction, in many practical cases, it is sufficient to keep the linear interaction term. The widely used electric dipole and quadrupole interaction operators are linear in the field variables.

In [32], a gauge-invariant evolution equation was derived for the same case of a single electron in an EM field, but a different Wigner quasi-distribution function ρ_w. If we use the same notation, we can write the equation as

$$\left(\not{\partial}^T + \frac{1}{m}\not{\mathbf{P}} \cdot \not{\nabla}^X\right) F_s = 0, \tag{40}$$

where

$$\not{\partial}^T = \frac{\partial}{\partial t} + q\mathbf{E} \cdot \overrightarrow{\nabla}_{\mathbf{P}} \operatorname{sinc}\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right), \tag{41}$$

$$\not{\mathbf{P}} = \mathbf{P} + \frac{\hbar q}{2}\left(\overrightarrow{\nabla}_{\mathbf{P}} \times \mathbf{B}\right) \operatorname{sinc}'\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right), \tag{42}$$

$$\not{\nabla}^X = \overrightarrow{\nabla}_{\mathbf{r}} - q\left(\overrightarrow{\nabla}_{\mathbf{P}} \times \mathbf{B}\right) \operatorname{sinc}\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right). \tag{43}$$

In this case, the right-hand side of the equation vanishes in contrast to Eq. (31) in Levanda and Fleurov [32], where there is an exponential function that includes derivatives of time, energy, momentum, and location. This simplification becomes obvious when we consider how ρ_w in Levanda and Fleurov [32], which uses four-vectors, is related to the Stratonovich function. The function ρ_w uses another variable ϵ, which represents the total energy of the electron. It can be shown that ρ_w can be transformed into F_s by integrating over ϵ, see Appendix C.3. The Stratonovich function can thus be seen as a marginal distribution of ρ_w, which explains the similar, simplified structure of the strong formulation of the GIWE. This construction resembles the relation between the non-equilibrium Green's function and the density matrix in quantum transport theory, where the energy-integrated Green's function yields the particle density in phase space [33].

Next, we will transform this equation into its integral form to establish a connection to the GIWE derived in Nedjalkov et al. [31], and compare the requirements for the EM field regarding regularity and asymptotic behavior.

4.2. Weak formulation

Before we can transform the strong formulation of the GIWE into the weak formulation, we first need to investigate the pseudo-differential operators to replace the nabla operator of $\mathbf{P}$ with $\mathbf{s}$. This can be achieved by using the following lemmas. Consider that the Fourier transform $\mathcal{F}$ of a function $f(\mathbf{s})$ actually gives us a function that depends on the wave vector $\mathbf{k}$, but since we use $\mathbf{P}$, we define the Fourier transform as

$$\mathcal{F}(f)(\mathbf{P}) := \sqrt{C} \int \mathrm{d}\mathbf{s} \exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{P} \cdot \mathbf{s}\right) f(\mathbf{s}). \tag{44}$$

Lemma 4.2. *Let $g \in C^\infty(\mathbb{R})$ be an entire function and $f \in L^1(\Omega), \Omega \subseteq \mathbb{R}^d$. For the Fourier transform $\mathcal{F} : f(\mathbf{s}) \mapsto \hat{f}(\mathbf{P})$ holds*

$$\mathcal{F}\left[g\left(\mathrm{i}\overleftarrow{\nabla}_{\mathbf{r}} \cdot \mathbf{s}\right) f(\mathbf{s})\right] = g\left(-\hbar \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right) \mathcal{F}[f(\mathbf{s})]. \tag{45}$$

The proof can be found in Appendix C.1. When we set g to sinc and sinc′, we get

$$\operatorname{sinc}\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right) \mathcal{F}(f) = \mathcal{F}\left[\operatorname{sinc}\left(\frac{\mathrm{i}}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \mathbf{s}\right) f\right]. \tag{46}$$

$$\operatorname{sinc}'\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right) \mathcal{F}(f) = -\mathcal{F}\left[\operatorname{sinc}'\left(\frac{\mathrm{i}}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \mathbf{s}\right) f\right], \tag{47}$$

By investigating the expansion of sin and cos on the right-hand side of (47)–(46), we can use a different representation, given in the following Lemma.

Lemma 4.3. *Let $g \in C^\infty(\mathbb{R}^n), n \in \mathbb{N}$ be an entire function. Then, there holds*

$$g(\mathbf{r}) \operatorname{sinc}\left(\frac{\mathrm{i}}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \mathbf{s}\right) = \frac{1}{2} \int_{-1}^{1} \mathrm{d}\tau g\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right), \tag{48}$$

$$g(\mathbf{r}) \operatorname{sinc}'\left(\frac{\mathrm{i}}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \mathbf{s}\right) = -\frac{\mathrm{i}}{2} \int_{-1}^{1} \mathrm{d}\tau g\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right) \tau. \tag{49}$$

The proof is shown in Appendix C.2. To transform the third term on the left-hand side and the right-hand side of (37), we make use of the convolution theorem: $\sqrt{C}\mathcal{F}(u) * \mathcal{F}(v) = \mathcal{F}(u \cdot v)$. For this purpose, these terms have to be expressed in the form of $\mathcal{F}(u \cdot v)$. According to (7), we set $v = \overline{\rho}$ so that $\mathcal{F}(v) = F_s$. To find u for the third term on the left of (37), we use Lemmas 4.2 and 4.3 to show that

$$\begin{aligned}
\mathcal{F}(u \cdot \overline{\rho}) &= \mathbf{F} \cdot \operatorname{sinc}\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right) \nabla_{\mathbf{P}} F_s \\
&= \mathbf{F} \cdot \operatorname{sinc}\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right) \nabla_{\mathbf{P}} \mathcal{F}(\overline{\rho}) \\
&= \mathcal{F}\left[\frac{1}{\mathrm{i}\hbar} \mathbf{F} \cdot \operatorname{sinc}\left(\frac{\mathrm{i}}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \mathbf{s}\right) \mathbf{s}\overline{\rho}\right] \\
&= \mathcal{F}\left[\frac{1}{2\mathrm{i}\hbar} \int_{-1}^{1} \mathrm{d}\tau \mathbf{F}\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right) \cdot \mathbf{s}\overline{\rho}\right].
\end{aligned} \tag{50}$$

Thus, u can be identified as

$$u(\mathbf{s}) = \frac{1}{2\mathrm{i}\hbar} \int_{-1}^{1} \mathrm{d}\tau \mathbf{F}\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right) \cdot \mathbf{s}. \tag{51}$$

Using (49) for the right-hand side of (37) yields

$$\begin{aligned}
\mathcal{F}(u \cdot \overline{\rho}) &= \left(\overrightarrow{\nabla}_{\mathbf{P}} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{r}} \operatorname{sinc}'\left(\frac{\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overrightarrow{\nabla}_{\mathbf{P}}\right) \mathcal{F}(\overline{\rho}) \\
&= \mathcal{F}\left[-\frac{1}{\mathrm{i}\hbar}\left(\mathbf{s} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{r}} \operatorname{sinc}'\left(\frac{\mathrm{i}}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \mathbf{s}\right) \overline{\rho}\right] \\
&= \mathcal{F}\left[\frac{1}{2\hbar} \int_{-1}^{1} \mathrm{d}\tau \left[\mathbf{s} \times \mathbf{B}\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right)\right] \tau \cdot \nabla_{\mathbf{r}} \overline{\rho}\right].
\end{aligned}$$

This gives

$$u(\mathbf{s}) = \frac{1}{2\hbar} \int_{-1}^{1} \mathrm{d}\tau \left[\mathbf{s} \times \mathbf{B}\left(\mathbf{r} + \frac{\mathbf{s}\tau}{2}\right)\right] \tau \cdot \nabla_{\mathbf{r}}. \tag{52}$$

The last term on the right-hand side of (37) is treated analogously.

$$\mathcal{F}(u\cdot\overline{\rho}) = \left(\overline{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\operatorname{sinc}\left(\frac{\hbar}{2}\overline{\nabla}_{\mathbf{r}}\cdot\overline{\nabla}_{\mathbf{P}}\right)\cdot\left(\overline{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\operatorname{sinc}'\left(\frac{\hbar}{2}\overline{\nabla}_{\mathbf{r}}\cdot\overline{\nabla}_{\mathbf{P}}\right)\mathcal{F}(\overline{\rho})$$
$$= \mathcal{F}\left[\frac{1}{\hbar^2}(\mathbf{s}\times\mathbf{B})\operatorname{sinc}\left(\frac{\mathrm{i}}{2}\overline{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)\cdot(\mathbf{s}\times\mathbf{B})\operatorname{sinc}'\left(\frac{\mathrm{i}}{2}\overline{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)\overline{\rho}\right]$$
$$= \mathcal{F}\left\{\frac{1}{\hbar^2}\int_{-1}^{1}\frac{\mathrm{d}\tau}{2}\left[\mathbf{s}\times\mathbf{B}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\cdot\int_{-1}^{1}\frac{\mathrm{d}\eta}{2\mathrm{i}}\left[\mathbf{s}\times\mathbf{B}\left(\mathbf{r}+\frac{\mathbf{s}\eta}{2}\right)\right]\eta\overline{\rho}\right\}.$$

In this case, we find

$$u(\mathbf{s}) = \frac{1}{4\mathrm{i}\hbar^2}\int_{-1}^{1}\mathrm{d}\tau\left[\mathbf{s}\times\mathbf{B}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\cdot\int_{-1}^{1}\mathrm{d}\eta\left[\mathbf{s}\times\mathbf{B}\left(\mathbf{r}+\frac{\mathbf{s}\eta}{2}\right)\right]\eta. \tag{53}$$

Applying the convolution theorem together with (51)–(53) to (37) yields

Theorem 4.4 (Weak formulation of the GIWE)**.** *Let $\tilde{\rho}_0$ be the density matrix at $t=0$ in an EM field $(\mathbf{E},\mathbf{B})$. For the Stratonovich function with the initial condition*

$$F_{\mathrm{s}}^{0}(\mathbf{P},\mathbf{r}) = C\int\mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+\frac{q}{2}\int_{-1}^{1}\mathrm{d}\tau\mathbf{A}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\right\}\tilde{\rho}_0(\mathbf{s},\mathbf{r}), \tag{54}$$

the evolution of F_{s} is given by

$$\frac{\partial F_{\mathrm{s}}}{\partial t}+\frac{\mathbf{P}}{m}\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}} = \int\mathrm{d}\mathbf{P}'L_{\mathrm{w}}(\mathbf{P},\mathbf{P}-\mathbf{P}',\mathbf{r})F_{\mathrm{s}}(\mathbf{P}',\mathbf{r})$$
$$+\int\mathrm{d}\mathbf{P}'\mathbf{M}_{\mathrm{w}1}(\mathbf{P}-\mathbf{P}',\mathbf{r})\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}}(\mathbf{P}',\mathbf{r})$$
$$+\int\mathrm{d}\mathbf{P}'M_{\mathrm{w}2}(\mathbf{P}-\mathbf{P}',\mathbf{r})F_{\mathrm{s}}(\mathbf{P}',\mathbf{r}), \tag{55}$$

where

$$L_{\mathrm{w}}(\mathbf{P}',\mathbf{P},\mathbf{r}) := -\frac{1}{2\mathrm{i}\hbar}C\int\mathrm{d}\mathbf{s}\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P}\right)\int_{-1}^{1}\mathrm{d}\tau\mathbf{F}\left(\mathbf{P}',\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\cdot\mathbf{s}, \tag{56}$$

$$\mathbf{M}_{\mathrm{w}1}(\mathbf{P},\mathbf{r}) := -\frac{q}{4m}C\int\mathrm{d}\mathbf{s}\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P}\right)\int_{-1}^{1}\mathrm{d}\tau\left[\mathbf{s}\times\mathbf{B}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\tau, \tag{57}$$

$$M_{\mathrm{w}2}(\mathbf{P},\mathbf{r}) := \frac{q^2}{8\mathrm{i}\hbar m}C\int\mathrm{d}\mathbf{s}\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P}\right)\int_{-1}^{1}\mathrm{d}\tau\left[\mathbf{s}\times\mathbf{B}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]$$
$$\cdot\int_{-1}^{1}\mathrm{d}\eta\left[\mathbf{s}\times\mathbf{B}\left(\mathbf{r}+\frac{\mathbf{s}\eta}{2}\right)\right]\eta. \tag{58}$$

It does not contradict any content in the paper, but might be an important clarification. Since we have used Lemma 4.3, the equivalence of the weak and strong formulations is only guaranteed if $\mathbf{E}$ and $\mathbf{B}$ are whole functions. As the functions (56)–(58) are defined as Fourier transforms of the fields, a sufficient condition for their existence is that the line integrals in L_{w} and $\mathbf{M}_{\mathrm{w}1}$ are $L^1(\mathbb{R}^3)$, and in $M_{\mathrm{w}2}$ are $L^2(\mathbb{R}^3)$. This poses different requirements to the EM field: The strong formulation imposes strong regularity conditions (smoothness), but does not constrain the asymptotic behavior, as the equation is local. In contrast, the integrability of the weak formulation relaxes the regularity requirements but introduces constraints to the asymptotic behavior. When we consider distributions for the EM field, like in the case of point charges, the weak formulation can easily be evaluated in the integrals. This is also possible for the strong formulation by using weak derivatives, which introduces integrals but preserves locality.

Theorem 4.4 is the same formulation as the one derived in Nedjalkov et al. [31], which also used the WS transform but started from the von Neumann equation. Thus, we can conclude the relations between the different formalisms: Starting from the Schrödinger equation, the von Neumann equation is derived. From there, the path splits into two, where one directly gives us the weak formulation of the GIWE. The other path leads to Moyal's formulation of the Wigner equation, where inserting the Hamiltonian for a charged particle into an EM field yields the strong formulation of the GDWE. The concatenation of the inverse Wigner and WS transform converts this equation into the strong formulation of the GIWE. The cycle is closed by transforming the strong into the weak formulation.

4.3. Conservation law for the probability current in EM fields

According to wave mechanics, the probability current of a charged particle in an EM field is given as

$$\mathbf{j} := -\frac{\mathrm{i}\hbar}{2m}[\psi^*\nabla\psi-\psi\nabla\psi^*]-\frac{q}{m}\psi^*\psi\mathbf{A}. \tag{59}$$

Since the corresponding expression for the current density in gauge-dependent Wigner mechanics is given as $\mathbf{j}(\mathbf{r}) = \int m^{-1}\mathbf{p}f_{\mathrm{w}}(\mathbf{p},\mathbf{r})\mathrm{d}\mathbf{p}$, it is natural to assume that this holds also in the gauge-invariant case, using F_{s}. Indeed, we can prove that

$$\mathbf{j}(\mathbf{r}) = \int\mathrm{d}\mathbf{P}\frac{\mathbf{P}}{m}F_{\mathrm{s}}(\mathbf{P},\mathbf{r}). \tag{60}$$

To show this, we need to calculate the derivative of the line integral of $\mathbf{A}$ at $\mathbf{s}=\mathbf{0}$, which we will apply later in (63).

$$\nabla_{\mathbf{s}}\left[\operatorname{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\right)\mathbf{s}\cdot\mathbf{A}(\mathbf{r})\right]\Big|_{\mathbf{s}=\mathbf{0}}$$
$$= \operatorname{sinc}'\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\right)\frac{\mathrm{i}}{2}\nabla_{\mathbf{r}}(\mathbf{s}\cdot\mathbf{A}(\mathbf{r}))+\operatorname{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\right)\mathbf{A}(\mathbf{r})\Big|_{\mathbf{s}=\mathbf{0}} = \mathbf{A}(\mathbf{r}). \tag{61}$$

We also calculate

$$\nabla_{\mathbf{s}}\tilde{\rho}(\mathbf{s},\mathbf{r})|_{\mathbf{s}=\mathbf{0}} = \nabla_{\mathbf{s}}\left[\psi\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right)\psi^*\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right]\Big|_{\mathbf{s}=\mathbf{0}}$$
$$= \frac{1}{2}[\nabla\psi(\mathbf{r})\psi^*(\mathbf{r})-\psi(\mathbf{r})\nabla\psi^*(\mathbf{r})], \tag{62}$$

which gives

$$\int\mathrm{d}\mathbf{P}\frac{\mathbf{P}}{m}F_{\mathrm{s}}(\mathbf{P},\mathbf{r})$$
$$= C\int\mathrm{d}\mathbf{P}\int\mathrm{d}\mathbf{s}\frac{\mathrm{i}\hbar}{m}\nabla_{\mathbf{s}}\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P}\right)\exp\left[\frac{q}{2\mathrm{i}\hbar}\mathbf{s}\cdot\int_{-1}^{1}\mathrm{d}\tau\mathbf{A}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\tilde{\rho}(\mathbf{s},\mathbf{r})$$
$$= -C\int\mathrm{d}\mathbf{P}\int\mathrm{d}\mathbf{s}\frac{\mathrm{i}\hbar}{m}\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P}\right)\nabla_{\mathbf{s}}\left\{\exp\left[\frac{q}{2\mathrm{i}\hbar}\mathbf{s}\cdot\int_{-1}^{1}\mathrm{d}\tau\mathbf{A}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\tilde{\rho}(\mathbf{s},\mathbf{r})\right\}$$
$$= -\int\mathrm{d}\mathbf{s}\delta(\mathbf{s})\frac{\mathrm{i}\hbar}{m}\exp\left[\frac{q}{2\mathrm{i}\hbar}\mathbf{s}\cdot\int_{-1}^{1}\mathrm{d}\tau\mathbf{A}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]$$
$$\left\{\frac{q}{\mathrm{i}\hbar}\nabla_{\mathbf{s}}\left[\mathbf{s}\cdot\operatorname{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\right)\mathbf{A}(\mathbf{r})\right]+\nabla_{\mathbf{s}}\tilde{\rho}(\mathbf{s},\mathbf{r})\right\}$$
$$= -\frac{\mathrm{i}\hbar}{2m}[\nabla\psi(\mathbf{r})\psi^*(\mathbf{r})-\psi(\mathbf{r})\nabla\psi^*(\mathbf{r})]-\frac{q}{m}\mathbf{A}(\mathbf{r}) = \mathbf{j}(\mathbf{r}). \tag{63}$$

We also find that the probability density can be calculated as

$$P(\mathbf{r}) = \int\mathrm{d}\mathbf{P}F_{\mathrm{s}}(\mathbf{P},\mathbf{r}). \tag{64}$$

Thus, the continuity equation can be rewritten as

$$\frac{\partial P}{\partial t}(\mathbf{r})+\nabla_{\mathbf{r}}\cdot\mathbf{j}(\mathbf{r}) = 0 \Leftrightarrow \int\mathrm{d}\mathbf{P}\left[\frac{\partial F_{\mathrm{s}}}{\partial t}(\mathbf{P},\mathbf{r})+\frac{\mathbf{P}}{m}\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}}(\mathbf{P},\mathbf{r})\right] = 0, \tag{65}$$

where the left-hand side of the GIWE can immediately be identified inside the integral. This implies that integrating the quantum correction terms over $\mathbf{P}$ yields zero, and the integration of the whole GIWE directly yields the continuity equation.

5. Conclusion

The Stratonovich formalism opens up possibilities for analyzing the kinetic behavior of quantum particles, as it uses the kinetic momentum instead of the canonical momentum, which is used in the standard Wigner theory. This enables experimental approbation of the theoretical results, as the kinetic momentum, being an observable, can be measured in an experiment, providing its mean value and distribution. To establish the Stratonovich formalism as a standalone description of a quantum system, the derivation of an evolution equation is a crucial step. The Weyl–Stratonovich transform defines a new function by replacing the canonical momentum with the kinetic momentum and changing the exponent of the Weyl transform. In this way, a gauge-invariant function, called the Stratonovich function, is introduced. The gauge-dependent and gauge-invariant theories are obtained from the density matrix and the corresponding von-Neumann evolution equation, without experience with the interrelationships between these theories. We demonstrate that Moyal's approach provides a comprehensive understanding of their connections. Moyal's formulation of the Wigner equation is transformed for the Stratonovich function, where all gauge variables $\mathbf{A}$ and ϕ are replaced by the field variables $\mathbf{E}$ and $\mathbf{B}$. This yields the strong formulation of the gauge-invariant Wigner equation, which is then transformed into the weak formulation. The two formulations pose different requirements to the regularity and asymptotic properties of the fields $\mathbf{E}$ and $\mathbf{B}$. However, we show that both equations are equivalent for a given field, allowing for a common formulation. Finally, it is demonstrated that integrating the gauge-invariant Wigner equations over momentum space yields the continuity equation, which states that the total probability is conserved during evolution.

However, the applicability of this formalism has to be investigated first. Important properties, such as space and time symmetry, boundedness, and evaluating the probability density function of the momentum by integrating over $\mathbf{r}$, do not directly translate to the Stratonovich function and thus need to be proven. Another interesting question arises regarding the definition of the Stratonovich function: Under which conditions is the Stratonovich function well-defined? Complications that potentially occur in non-convex domains might require modifications or generalizations of the definition, which needs to be investigated in future research.

CRediT authorship contribution statement

Clemens Etl: Writing – original draft, Methodology, Investigation, Conceptualization; **Mauro Ballicchia:** Writing – review & editing, Investigation; **Mihail Nedjalkov:** Writing – review & editing, Investigation, Conceptualization; **Hans Kosina:** Writing – review & editing, Supervision, Project administration, Investigation, Funding acquisition.

Data availability

No data was used for the research described in the article.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Weyl–Stratonovich transform

A.1. Gauge-dependence of F_w

$$\begin{aligned}F'_\mathrm{w}(\mathbf{P},\mathbf{r},t) &= C\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot[\mathbf{P}+q\mathbf{A}'(\mathbf{r})]\right\}\tilde{\rho}'(\mathbf{s},\mathbf{r})\\ &= C\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot[\mathbf{P}+q\mathbf{A}(\mathbf{r})+q\nabla\chi(\mathbf{r})]\right\}\\ &\quad \exp\left\{-\frac{1}{\mathrm{i}\hbar}q\left[\chi\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right)-\chi\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right]\right\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\ &\neq C\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot[\mathbf{P}+q\mathbf{A}(\mathbf{r})]\right\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\ &= F_\mathrm{w}(\mathbf{P},\mathbf{r},t).\end{aligned} \tag{A.1}$$

A.2. Proof for Lemma 3.2

Proof. First, we transform

$$\begin{aligned}\chi\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right)-\chi\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right) &= \int_{-1}^{1}\mathrm{d}\tau\frac{\mathrm{d}\chi}{\mathrm{d}\tau}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\\ &= \int_{-1}^{1}\mathrm{d}\tau\frac{\mathbf{s}}{2}\cdot\nabla\chi\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right),\end{aligned}$$

which is used to show that

$$\begin{aligned}&F'_\mathrm{s}(\mathbf{P},\mathbf{r}) =\\ &C\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+\frac{q}{2}\int_{-1}^{1}\mathrm{d}\tau\mathbf{A}'\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\right\}\tilde{\rho}'(\mathbf{s},\mathbf{r})\\ &= C\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+\frac{q}{2}\int_{-1}^{1}\mathrm{d}\tau\mathbf{A}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right.\right.\\ &\quad\left.\left.+\nabla\chi\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\right\}\exp\left\{-\frac{q}{\mathrm{i}\hbar}\left[\chi\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right)-\chi\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right]\right\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\ &= C\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+\frac{q}{2}\int_{-1}^{1}\mathrm{d}\tau\mathbf{A}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right.\right.\\ &\quad\left.\left.+\nabla\chi\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\right\}\exp\left\{-\frac{q}{2\mathrm{i}\hbar}\int_{-1}^{1}\mathrm{d}\tau\mathbf{s}\cdot\nabla\chi\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\ &= C\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+\frac{q}{2}\int_{-1}^{1}\mathrm{d}\tau\mathbf{A}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\right]\right\}\tilde{\rho}(\mathbf{s},\mathbf{r}) = F_\mathrm{s}(\mathbf{P},\mathbf{r}).\end{aligned}$$

□

A.3. Properties of $\mathcal{T}$: derivatives

Proof. We can replace the line integral in the exponent of F_s by sinc using (48), so that

$$F_\mathrm{s}(\mathbf{P},\mathbf{r}) = C\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+q\,\mathrm{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_\mathbf{r}\right)\mathbf{A}(\mathbf{r})\right]\right\}\tilde{\rho}(\mathbf{s},\mathbf{r}). \tag{A.2}$$

Note that the nabla operator in the sinc-function only acts on $\mathbf{A}$. The proof for $\partial/\partial t$ and $\nabla_\mathbf{r}$ is the same. Therefore, we write α, which could stand for x, y, z or t.

$$\begin{aligned}\frac{\partial F_\mathrm{s}}{\partial\alpha} &= \int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot[\mathbf{P}+q\,\mathrm{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_\mathbf{r}\right)\mathbf{A}(\mathbf{r},t)]\right\}\frac{\partial\tilde{\rho}}{\partial\alpha}(\mathbf{s},\mathbf{r})\\ &\quad+\int \mathrm{d}\mathbf{s}\frac{\partial}{\partial\alpha}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot[\mathbf{P}+q\,\mathrm{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_\mathbf{r}\right)\mathbf{A}(\mathbf{r},t)]\right\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\ &= \mathcal{T}\left(\frac{\partial f_\mathrm{w}}{\partial\alpha}\right)+\frac{q}{\mathrm{i}\hbar}\int \mathrm{d}\mathbf{s}\,\mathrm{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_\mathbf{r}\right)\left[\mathbf{s}\cdot\frac{\partial\mathbf{A}}{\partial\alpha}(\mathbf{r},t)\right]\\ &\quad\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+q\,\mathrm{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_\mathbf{r}\right)\mathbf{A}(\mathbf{r},t)\right]\right\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\ &= \mathcal{T}\left(\frac{\partial f_\mathrm{w}}{\partial\alpha}\right)+q\left[\overline{\nabla}_\mathbf{P}\cdot\frac{\partial\mathbf{A}}{\partial\alpha}(\mathbf{r},t)\right]\mathrm{sinc}\left(\frac{\hbar}{2}\overline{\nabla}_\mathbf{r}\cdot\overline{\nabla}_\mathbf{P}\right)\\ &\quad\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+q\,\mathrm{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_\mathbf{r}\right)\mathbf{A}(\mathbf{r},t)\right]\right\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\ &= \mathcal{T}\left(\frac{\partial f_\mathrm{w}}{\partial\alpha}\right)+q\left[\overline{\nabla}_\mathbf{P}\cdot\frac{\partial\mathbf{A}}{\partial\alpha}(\mathbf{r},t)\right]\mathrm{sinc}\left(\frac{\hbar}{2}\overline{\nabla}_\mathbf{r}\cdot\overline{\nabla}_\mathbf{P}\right)F_\mathrm{s},\end{aligned}$$

where we have used Lemma 4.2. The same approach cannot be used for $\nabla_\mathbf{p}$ as the exponential function of f_w depends on $\mathbf{p}$. Thus, we show

$$\begin{aligned}\nabla_\mathbf{P}F_\mathrm{s} &= \nabla_\mathbf{P}\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+q\,\mathrm{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_\mathbf{r}\right)\mathbf{A}(\mathbf{r},t)\right]\right\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\ &= \frac{1}{\mathrm{i}\hbar}\int \mathrm{d}\mathbf{s}\exp\left\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\left[\mathbf{P}+q\,\mathrm{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_\mathbf{r}\right)\mathbf{A}(\mathbf{r},t)\right]\right\}\mathbf{s}\tilde{\rho}(\mathbf{s},\mathbf{r})\\ &= \mathcal{T}\left[\frac{1}{\mathrm{i}\hbar}\int \mathrm{d}\mathbf{s}\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{p}\right)\mathbf{s}\tilde{\rho}(\mathbf{s},\mathbf{r})\right]\\ &= \mathcal{T}\left[\nabla_\mathbf{p}\int \mathrm{d}\mathbf{s}\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{p}\right)\tilde{\rho}(\mathbf{s},\mathbf{r})\right]\\ &= \mathcal{T}[\nabla_\mathbf{p}f_\mathrm{w}].\end{aligned}$$

□

A.4. Properties of $\mathcal{T}$: $\mathbf{p}f_{\mathrm{w}}$

First, we calculate the inverse Weyl transformation of $\mathbf{p}f_{\mathrm{w}}$, using $C\int \mathrm{d}\mathbf{p}\, e^{\mathrm{i}/\hbar(\mathbf{s}-\mathbf{s}')\cdot\mathbf{p}} = \delta(\mathbf{s}-\mathbf{s}')$.

$$
\begin{aligned}
\mathcal{W}^{-1}(\mathbf{p}f_{\mathrm{w}}) &= C\int \mathrm{d}\mathbf{p}\exp\Big(-\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{p}\Big)\mathbf{p}\int \mathrm{d}\mathbf{s}'\exp\Big(\frac{1}{\mathrm{i}\hbar}\mathbf{s}'\cdot\mathbf{p}\Big)\tilde{\rho}(\mathbf{s}',\mathbf{r})\\
&= \mathrm{i}\hbar\int \mathrm{d}\mathbf{s}' C\int \mathrm{d}\mathbf{p}\nabla_{\mathbf{s}'}\Big(e^{-\frac{1}{\mathrm{i}\hbar}(\mathbf{s}-\mathbf{s}')\cdot\mathbf{p}}\Big)\tilde{\rho}(\mathbf{s}',\mathbf{r})\\
&= -\mathrm{i}\hbar\int \mathrm{d}\mathbf{s}' C\int \mathrm{d}\mathbf{p}\, e^{-\frac{1}{\mathrm{i}\hbar}(\mathbf{s}-\mathbf{s}')\cdot\mathbf{p}}\nabla_{\mathbf{s}'}\tilde{\rho}(\mathbf{s}',\mathbf{r})\\
&= -\mathrm{i}\hbar\int \mathrm{d}\mathbf{s}'\delta(\mathbf{s}-\mathbf{s}')\nabla_{\mathbf{s}'}\tilde{\rho}(\mathbf{s}',\mathbf{r})\\
&= -\mathrm{i}\hbar\nabla_{\mathbf{s}}\tilde{\rho}(\mathbf{s},\mathbf{r}).
\end{aligned}
$$

Next, with the help of Lemma 4.2, we apply the WS transform, which yields

$$
\begin{aligned}
\mathcal{T}(\mathbf{p}f_{\mathrm{w}}) &= \mathcal{S}\big(-\mathrm{i}\hbar\nabla_{\mathbf{s}}\tilde{\rho}(\mathbf{s},\mathbf{r})\big)\\
&= -\mathrm{i}\hbar C\int \mathrm{d}\mathbf{s}\exp\Big\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\Big[\mathbf{P}+q\,\mathrm{sinc}\Big(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\Big)\mathbf{A}(\mathbf{r})\Big]\Big\}\nabla_{\mathbf{s}}\tilde{\rho}(\mathbf{s},\mathbf{r})\\
&= \mathrm{i}\hbar C\int \mathrm{d}\mathbf{s}\nabla_{\mathbf{s}}\exp\Big\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\Big[\mathbf{P}+q\,\mathrm{sinc}\Big(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\Big)\mathbf{A}(\mathbf{r})\Big]\Big\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\
&= C\int \mathrm{d}\mathbf{s}\Big[\mathbf{P}+q\,\mathrm{sinc}\Big(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\Big)\mathbf{A}(\mathbf{r})+\frac{\mathrm{i}q}{2}\,\mathrm{sinc}'\Big(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\Big)\nabla_{\mathbf{r}}(\mathbf{s}\cdot\mathbf{A})\Big]\\
&\quad\exp\Big\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\Big[\mathbf{P}+q\,\mathrm{sinc}\Big(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\Big)\mathbf{A}(\mathbf{r})\Big]\Big\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\
&= C\int \mathrm{d}\mathbf{s}\Big[\mathbf{P}+q\mathbf{A}(\mathbf{r})\,\mathrm{sinc}\Big(-\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)-\\
&\quad\frac{q\hbar}{2}\nabla_{\mathbf{r}}(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A})\,\mathrm{sinc}'\Big(-\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\Big]\\
&\quad\exp\Big\{\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\Big[\mathbf{P}+q\,\mathrm{sinc}\Big(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}\Big)\mathbf{A}(\mathbf{r})\Big]\Big\}\tilde{\rho}(\mathbf{s},\mathbf{r})\\
&= \Big[\mathbf{P}+q\mathbf{A}(\mathbf{r})\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)+\frac{q\hbar}{2}\nabla_{\mathbf{r}}(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A})\,\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\Big]\\
&\quad F_{\mathrm{s}}(\mathbf{s},\mathbf{r})
\end{aligned}
$$

Appendix B. Gauge-invariant WE

B.1. Derivation of the strong formulation

We start with the strong formulation of the GDWE and expand the brackets:

$$
\begin{aligned}
&\frac{\partial f_{\mathrm{w}}}{\partial t}+\frac{2q}{\hbar m}\mathbf{p}\cdot\mathbf{A}\sin\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}\Big)f_{\mathrm{w}}-\frac{q^2}{\hbar m}\mathbf{A}^2\sin\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}\Big)f_{\mathrm{w}}\\
&-\frac{2q}{\hbar}\phi\sin\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}\Big)f_{\mathrm{w}}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}}f_{\mathrm{w}}-\frac{q}{m}\mathbf{A}\cdot\cos\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}\Big)\nabla_{\mathbf{r}}f_{\mathrm{w}}=0.
\end{aligned}
\tag{B.1}
$$

Then we transform each term as described in step 1. The terms, for which we will use the identity (32) from step 2, and $\mathbf{E}=-\nabla_{\mathbf{r}}\phi-\partial\mathbf{A}/\partial t$, are labeled with Roman numbers to identify which of them are combined. We also apply the identities (34) and (35) of step 4: The transformation of the time derivative yields

$$
\mathcal{T}\Big(\frac{\partial f_{\mathrm{w}}}{\partial t}\Big)\overset{(28)}{=}\frac{\partial F_{\mathrm{s}}}{\partial t}-q\frac{\partial\mathbf{A}}{\partial t}\cdot\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\nabla_{\mathbf{P}}F_{\mathrm{s}}^{\mathrm{I.a}}.
\tag{B.2}
$$

By using the multiplicativity of the transformation, the term containing $\mathbf{p}\cdot\mathbf{A}$ transforms as

$$
\begin{aligned}
&\mathcal{T}\Big[\frac{2q}{\hbar m}\mathbf{p}\cdot\mathbf{A}\sin\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}\Big)f_{\mathrm{w}}\Big]\\
&=\frac{2q}{\hbar m}\mathbf{A}\sin\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}'\Big)\cdot\mathcal{T}(\mathbf{p}f_{\mathrm{w}}')\\
&\overset{(31)}{=}\frac{2q}{\hbar m}\Big[\mathbf{P}+\frac{q\hbar}{2}\nabla_{\mathbf{r}}\Big(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A}\Big)\,\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)+q\mathbf{A}\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\Big]\cdot\\
&\quad\mathbf{A}\sin\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)F_{\mathrm{s}}\\
&\overset{(34)}{=}\frac{q}{m}\mathbf{P}\cdot\mathbf{A}\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)F_{\mathrm{s}}^{\mathrm{II.a}}\\
&\quad+\frac{q^2}{m}\mathbf{A}\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\cdot\mathbf{A}\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)F_{\mathrm{s}}^{\mathrm{III.a}}\\
&\quad+\frac{q^2\hbar}{2m}\nabla_{\mathbf{r}}\Big(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A}\Big)\,\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\\
&\quad\cdot\mathbf{A}\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)F_{\mathrm{s}}^{\mathrm{IV.a}}.
\end{aligned}
\tag{B.3}
$$

For the term containing $\mathbf{A}^2$ we use (33) from step 3, which gives us

$$
\begin{aligned}
&\mathcal{T}\Big[-\frac{q^2}{\hbar m}\mathbf{A}^2\sin\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}\Big)f_{\mathrm{w}}\Big]\\
&\overset{(33)}{=}-\frac{2q^2}{\hbar m}\Big[\mathbf{A}\sin\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\Big]\cdot\Big[\mathbf{A}\cos\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\Big]F_{\mathrm{s}}\\
&\overset{(34),(35)}{=}-\frac{q^2}{m}\mathbf{A}\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\\
&\quad\cdot\Big[\mathbf{A}\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)+\frac{\hbar}{2}\mathbf{A}\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\,\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\Big]F_{\mathrm{s}}\\
&=-\frac{q^2}{m}\mathbf{A}\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\cdot\mathbf{A}\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)F_{\mathrm{s}}^{\mathrm{III.b}}\\
&\quad-\frac{q^2\hbar}{2m}\mathbf{A}\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\cdot\mathbf{A}\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\\
&\quad\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)F_{\mathrm{s}}^{\mathrm{IV.b}}.
\end{aligned}
\tag{B.4}
$$

The term containing the scalar potential ϕ becomes

$$
\mathcal{T}\Big[-\frac{2q}{\hbar}\phi\sin\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}\Big)f_{\mathrm{w}}\Big]\overset{(34)}{=}-q\nabla_{\mathbf{r}}\phi\cdot\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\nabla_{\mathbf{P}}F_{\mathrm{s}}^{\mathrm{I.b}}.
\tag{B.5}
$$

The term containing $\nabla_r f_w$ represents a diffusion term and transforms as

$$
\begin{aligned}
&\mathcal{T}\Big\{\Big[\frac{\mathbf{p}}{m}-\frac{q}{m}\mathbf{A}\cos\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\Big]\cdot\nabla_{\mathbf{r}}f_{\mathrm{w}}\Big\}\\
&\overset{(31),(35)}{=}\Big[\frac{\mathbf{P}}{m}+\frac{q\hbar}{2m}\nabla_{\mathbf{r}}\Big(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A}\Big)\,\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\\
&\quad-\frac{q\hbar}{2m}\mathbf{A}\Big(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\,\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\Big]\cdot\mathcal{T}(\nabla_{\mathbf{r}}f_{\mathrm{w}})\\
&\overset{(29),(32)}{=}\Big[\frac{\mathbf{P}}{m}+\frac{q\hbar}{2m}\Big(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\Big)\,\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\Big]\\
&\quad\cdot\Big[\nabla_{\mathbf{r}}F_{\mathrm{s}}-q\nabla_{\mathbf{r}}(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A})\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)F_{\mathrm{s}}\Big]\\
&=\frac{\mathbf{P}}{m}\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}}+\frac{q\hbar}{2m}\Big(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\Big)\,\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}}\\
&\quad-\frac{q}{m}\mathbf{P}\cdot\nabla_{\mathbf{r}}(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A})\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)F_{\mathrm{s}}^{\mathrm{II.b}}\\
&\quad-\frac{q^2\hbar}{2m}\Big(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\Big)\,\mathrm{sinc}'\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\cdot\nabla_{\mathbf{r}}(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A})\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\\
&\quad F_{\mathrm{s}}^{\mathrm{IV.c}}.
\end{aligned}
\tag{B.6}
$$

In the above equation, the term $q/m\mathbf{A}\,\mathrm{sinc}(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}})$ appeared twice and has been canceled out.

Now we can sum up all terms as described in step 5. The two terms labeled with III add up to zero. For I we use $\mathbf{E}=-\nabla\phi-\partial\mathbf{A}/\partial t$, which yields

$$
\begin{aligned}
&\mathrm{I}=\mathrm{I.a}+\mathrm{I.b}\\
&=-q\frac{\partial\mathbf{A}}{\partial t}\cdot\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\nabla_{\mathbf{P}}F_{\mathrm{s}}-q\nabla_{\mathbf{r}}\phi\cdot\,\mathrm{sinc}\Big(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\Big)\nabla_{\mathbf{P}}F_{\mathrm{s}}
\end{aligned}
$$

$$= q\mathbf{E}\cdot\ \mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\nabla_{\mathbf{P}}F_{\mathrm{s}}. \tag{B.7}$$

Using (32) and the scalar triple product formula transforms II as

$$\begin{aligned}\mathrm{II} &= \mathrm{II.a}+\mathrm{II.b}\\ &= \frac{q}{m}\mathbf{P}\cdot\mathbf{A}\left(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}\\ &\quad-\frac{q}{m}\mathbf{P}\cdot\nabla_{\mathbf{r}}(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A})\,\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}\\ &=-\frac{q}{m}\mathbf{P}\cdot\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}\\ &=\frac{q}{m}(\mathbf{P}\times\mathbf{B})\cdot\overrightarrow{\nabla}_{\mathbf{P}}\,\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}.\end{aligned} \tag{B.8}$$

Also for IV (32) is used to evaluate

$$\begin{aligned}\mathrm{IV} &= \mathrm{IV.a}+\mathrm{IV.b}+\mathrm{IV.c}\\ &=\frac{q^2\hbar}{2m}\nabla_{\mathbf{r}}\left(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\mathbf{A}\left(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}\\ &\quad-\frac{q^2\hbar}{2m}\mathbf{A}\left(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\mathbf{A}\left(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}\\ &\quad-\frac{q^2\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\nabla_{\mathbf{r}}(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A})\,\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}\\ &=\frac{q^2\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\mathbf{A}\left(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}\\ &\quad-\frac{q^2\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\nabla_{\mathbf{r}}(\overrightarrow{\nabla}_{\mathbf{P}}\cdot\mathbf{A})\,\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}\\ &=-\frac{q^2\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}.\end{aligned} \tag{B.9}$$

Adding up all terms gives

$$\begin{aligned}&\frac{\partial F_{\mathrm{s}}}{\partial t}+\frac{\mathbf{P}}{m}\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}}+\frac{q\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}}+\mathrm{I}+\mathrm{II}+\mathrm{IV}=0\\ \Leftrightarrow\ &\frac{\partial F_{\mathrm{s}}}{\partial t}+\frac{\mathbf{P}}{m}\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}}+\frac{q\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}}\\ &+q\mathbf{E}\cdot\ \mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\nabla_{\mathbf{P}}F_{\mathrm{s}}+\frac{q}{m}(\mathbf{P}\times\mathbf{B})\cdot\overrightarrow{\nabla}_{\mathbf{P}}\,\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}\\ &-\frac{q^2\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)F_{\mathrm{s}}=0.\end{aligned} \tag{B.10}$$

To simplify the remaining terms according to step 6, we factor out the sinc' expression and introduce the Lorentz force $\mathbf{F}=q\mathbf{E}+qm^{-1}\mathbf{P}\times\mathbf{B}$, which yields

$$\begin{aligned}&\frac{\partial F_{\mathrm{s}}}{\partial t}+\frac{\mathbf{P}}{m}\cdot\nabla_{\mathbf{r}}F_{\mathrm{s}}+\mathbf{F}\cdot\ \mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\nabla_{\mathbf{P}}F_{\mathrm{s}}\\ &=-\frac{q\hbar}{2m}\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}'\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\cdot\left[\overleftarrow{\nabla}_{\mathbf{r}}-q\left(\overrightarrow{\nabla}_{\mathbf{P}}\times\mathbf{B}\right)\mathrm{sinc}\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\right]F_{\mathrm{s}}.\end{aligned} \tag{B.11}$$

Appendix C. GDWE

C.1. Proof for Lemma 4.2

Proof. As g is analytic, we can write it as $g(x)=\sum_{n=0}^{\infty}a_nx^n$. We can also exchange the limit of the expansion with the integral, due to its uniform convergence. For a pseudo-differential operator of a polynomial $p(D)=\sum_\alpha a_\alpha D^\alpha$, $D^\alpha=(-\mathrm{i}\hbar\partial_1)^{\alpha_1}\dots(-\mathrm{i}\hbar\partial_n)^{\alpha_n}$ holds

$$\mathcal{F}[p(\mathbf{s})f(\mathbf{s})]=p(D)\mathcal{F}[f(\mathbf{s})].$$

This is used to show that

$$\begin{aligned}\mathcal{F}\left[g\left(\mathrm{i}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)f(\mathbf{s})\right]&=\sqrt{C}\int\mathrm{d}\mathbf{s}\,g\left(\mathrm{i}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P}\right)f(\mathbf{s})\\ &=\sqrt{C}\int\mathrm{d}\mathbf{s}\sum_{n=0}^{\infty}a_n\left(\mathrm{i}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)^n\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P}\right)f(\mathbf{s})\\ &=\sqrt{C}\lim_{N\to\infty}\int\mathrm{d}\mathbf{s}\sum_{n=0}^{N}a_n\left(\mathrm{i}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)^n\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P}\right)f(\mathbf{s})\\ &=\sqrt{C}\int\mathrm{d}\mathbf{s}\sum_{n=0}^{\infty}a_n\left(-\hbar\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)^n\exp\left(\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P}\right)f(\mathbf{s})\\ &=\sqrt{C}g\left(-\hbar\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{P}}\right)\mathcal{F}(f)(\mathbf{P}).\end{aligned}$$

□

C.2. Proof for Lemma 4.3

Proof. Using a Taylor expansion for τ yields

$$\begin{aligned}\int_{-1}^{1}\mathrm{d}\tau g\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)&=\int_{-1}^{1}\mathrm{d}\tau\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mathbf{s}\tau}{2}\cdot\nabla\right)^n g(\mathbf{r})\\ &=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{2^n}(\mathbf{s}\cdot\nabla)^n g(\mathbf{r})\frac{1}{n+1}\left(1^{n+1}-(-1)^{n+1}\right)\\ &=2\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\frac{1}{2^{2n}}(\mathbf{s}\cdot\nabla)^{2n}g(\mathbf{r})\\ &=2\,\mathrm{sinc}\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla\right)g(\mathbf{r}).\end{aligned}$$

Similarly, we can show that

$$\cos\left(\frac{\mathrm{i}\mathbf{s}}{2}\cdot\nabla\right)g(\mathbf{r})=\frac{1}{2}\left[g\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right)+g\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right],$$

which we use together with (35) and integration by parts to show that

$$\begin{aligned}g(\mathbf{r})\left(\frac{\mathrm{i}}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)\mathrm{sinc}'\left(\frac{\mathrm{i}}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)&=g(\mathbf{r})\cos\left(\frac{\mathrm{i}}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)-g(\mathbf{r})\,\mathrm{sinc}\left(\frac{\mathrm{i}}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)\\ &=\frac{1}{2}\left[g\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right)+g\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right]-\frac{1}{2}\int_{-1}^{1}\mathrm{d}\tau g\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\\ &=\frac{1}{2}\left[g\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\tau\right]_{\tau=-1}^{1}-\frac{1}{2}\int_{-1}^{1}\mathrm{d}\tau g\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\\ &=\frac{1}{2}\int_{-1}^{1}\mathrm{d}\tau\frac{\mathrm{d}g}{\mathrm{d}\tau}\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\tau\\ &=\frac{1}{2}\int_{-1}^{1}\mathrm{d}\tau\frac{\mathbf{s}}{2}\cdot\nabla_{\mathbf{r}}g\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\tau\\ &=-\frac{\mathrm{i}}{2}\left(\frac{\mathrm{i}}{2}\mathbf{s}\cdot\nabla_{\mathbf{r}}\right)\int_{-1}^{1}\mathrm{d}\tau g\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\tau.\end{aligned}$$

Therefore it holds

$$g(\mathbf{r})\,\mathrm{sinc}'\left(\frac{\mathrm{i}}{2}\overleftarrow{\nabla}_{\mathbf{r}}\cdot\mathbf{s}\right)=-\frac{\mathrm{i}}{2}\int_{-1}^{1}\mathrm{d}\tau g\left(\mathbf{r}+\frac{\mathbf{s}\tau}{2}\right)\tau.$$

□

C.3. Connection between ρ_{w} and F_{s}

In [32], a gauge-invariant Wigner operator is defined,

$$\hat{\rho}_{\mathrm{w}}(P,X):=\frac{C}{2\pi}\int\mathrm{d}^4y\exp\left[\frac{1}{\mathrm{i}\hbar}Py+\frac{q}{\mathrm{i}\hbar}\int_{-1/2}^{1/2}y^\mu A_\mu(X+sy)\mathrm{d}s\right]\hat{\rho}(y,X), \tag{C.1}$$

where index notation is used, and the 4-vectors are defined as $P=(\mathbf{P},\epsilon)$, $X=(\mathbf{r},t)$, $y=(\mathbf{s},\tau)$, $A=(\mathbf{A},\phi)$. This Wigner operator is connected to F_{s} via

$$\begin{aligned}&\int\mathrm{d}\epsilon\hat{\rho}_{\mathrm{w}}(P,X)\\ &=\frac{C}{2\pi}\int\mathrm{d}\epsilon\int\mathrm{d}^4y\hat{\rho}(y,X)\exp\left[\frac{1}{\mathrm{i}\hbar}Py+\frac{q}{\mathrm{i}\hbar}\int_{-1/2}^{1/2}y^\mu A_\mu(X+sy)\mathrm{d}s\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{C}{2\pi} \int \mathrm{d}\epsilon \int \mathrm{d}\tau \int \mathrm{d}s\, \hat{\rho}\left(\mathbf{r} + \frac{\mathbf{s}}{2}, t + \frac{\tau}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t - \frac{\tau}{2}\right) \\
&\quad \exp\left[\frac{1}{\mathrm{i}\hbar}(\mathbf{s}\cdot\mathbf{P} + \tau\epsilon) + \frac{q}{\mathrm{i}\hbar}\int_{-1}^{1} \mathrm{d}\gamma\, \mathbf{s}\cdot\mathbf{A}\left(\mathbf{r} + \frac{\mathbf{s}\gamma}{2}, t + \frac{\tau\gamma}{2}\right) + \tau\phi\left(\mathbf{r} + \frac{\mathbf{s}\gamma}{2}, t + \frac{\tau\gamma}{2}\right)\right] \\
&= C \int \mathrm{d}\tau \int \mathrm{d}\mathbf{s}\,\delta(\tau)\hat{\rho}\left(\mathbf{r} + \frac{\mathbf{s}}{2}, t + \frac{\tau}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t - \frac{\tau}{2}\right) \\
&\quad \exp\left[\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P} + \frac{q}{\mathrm{i}\hbar}\int_{-1}^{1} \mathrm{d}\gamma\, \mathbf{s}\cdot\mathbf{A}\left(\mathbf{r} + \frac{\mathbf{s}}{2}, t + \frac{\tau}{2}\right) + \tau\phi\left(\mathbf{r} + \frac{\mathbf{s}\gamma}{2}, t + \frac{\tau\gamma}{2}\right)\right] \\
&= C \int \mathrm{d}\mathbf{s} \exp\left[\frac{1}{\mathrm{i}\hbar}\mathbf{s}\cdot\mathbf{P} + \frac{q}{\mathrm{i}\hbar}\int_{-1}^{1} \mathrm{d}\gamma\, \mathbf{s}\cdot\mathbf{A}\left(\mathbf{r} + \frac{\mathbf{s}\gamma}{2}, t\right)\right]\hat{\rho}\left(\mathbf{r} + \frac{\mathbf{s}}{2}, t, \mathbf{r} - \frac{\mathbf{s}}{2}, t\right) \\
&= F_{\mathrm{s}}(\mathbf{P}, \mathbf{r}, t),
\end{aligned}
\tag{C.2}
$$

which shows that F_{s} is a marginal distribution of $\hat{\rho}_{\mathrm{w}}$.

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